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The symmetry of M-theories

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Abstract: We consider the Cartan subalgebra of any very extended algebra $\mathcal{G}^{+++}$ where $\mathcal{G}$ is a simple Lie algebra and let the parameters be space-time fields. These are identified with diagonal metrics and dilatons. Using the properties of the algebra, we find that for all very extensions $\mathcal{G}^{+++}$ of simple Lie algebras there are theories of gravity and matter, which admit classical solutions carrying representations of the Weyl group of $\mathcal{G}^{+++}$. We also identify the T and S-dualities of superstrings and of the bosonic string with Weyl reflections and outer automorphisms of well-chosen very extended algebras and we exhibit specific features of the very extensions. We take these results as indication that very extended algebras underlie symmetries of any consistent theory of gravity and matter, and might encode basic information for the construction of such theory.

Keywords: M-Theory, String Duality, Space-Time Symmetries, Classical Theories of Gravity.

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Contents

1. Introduction

There is a widespread belief that supersymmetry and string theory are essential ingredients in a single unified theory of physics. However, supergravity theories do not provide a consistent theory of quantum gravity and there exists no satisfactory theory of strings in the sense that we can calculate, even as a matter of principle, non-perturbative effects in non-trivial backgrounds. Much of what we know about superstring theory outside perturbation theory has been derived from the properties of the maximal supergravity theories in ten dimensions, namely the IIA supergravity [1, 2, 3] and IIB supergravity [4, 5, 6] theories as well as the type-I supergravity coupled to the Yang-Mills theory [7]. These theories are essentially determined by the type of supersymmetry they possess and hence they are the complete low energy effective actions of the superstring theories with the corresponding space-time supersymmetry. Although the existence of a unique supergravity theory in eleven dimensions [8] has been known for many years, it is only relatively recently that it has been conjectured [9, 10] to be a particular limit of a fully quantum theory called M-theory which would encompass all superstring theories. Very little is known for sure about M-theory, but it is thought to contain, in particular limits and at low energies, all the supergravity theories mentioned above.

JHEP09(2003)020

It is our feeling that essential elements are lacking in the quest for a unified theory of quantised gravity and matter. Bringing to light the symmetries that lie beneath the present attempts to such unification might help uncover those essential elements, which could actually hide in the symmetry structure itself. These hopes have been encouraged by some developments in recent years that have argued that certain types of Kac-Moody algebras occur in supergravity theories. Based on the formulation of eleven dimensional supergravity as a non-linear realisation [11], it has been conjectured that an extension of this theory possesses a rank eleven Kac-Moody algebra called E_{11} [12]. Furthermore, it has been argued that a rank twenty-seven Kac-Moody algebra, called K_{27} underlies the closed bosonic string in twenty-six dimensions [12] and that even pure gravity in D dimensions can be extended so as to possess a Kac-Moody algebra of rank D [13]. A systematic account of symmetries in supergravity and string theories is given at the end of this section.

One interesting aspect of the proposed Kac-Moody symmetries for M-theory, for the closed bosonic string and for gravity, is that they are all Kac-Moody algebras of a special type. Given any finite-dimensional simple Lie algebra $\mathcal{G}$, there is a well-known procedure for constructing a corresponding affine algebra $\mathcal{G}^+$ by adding a node to the Dynkin diagram in a certain way which is related to the properties of the highest root of $\mathcal{G}$. One may also further increase by one the rank of the algebra $\mathcal{G}^+$ by adding to the Dynkin diagram a further node that is attached to the affine node by a single line [14]. This is called the overextension $\mathcal{G}^{++}$. The very extension, also called triple extension in this paper, denoted $\mathcal{G}^{+++}$, is found by adding yet another node to the Dynkin diagram that is attached to the overextended node by one line [15]. The rank of very extended exceeds by three the rank of the finite-dimensional simple Lie algebra $\mathcal{G}$ from which one started. Some of the properties of very extended algebras including their roots and weights are given in [15]. Thus M-theory, the closed bosonic string and gravity are associated with the very extension of E_8, and of the D and A series of finite simple Lie algebras respectively.

It is tempting to suppose that there are also theories associated with the other finite-dimensional simple Lie algebras and that they possess symmetries that are the corresponding triple extensions $\mathcal{G}^{+++}$ of these algebras. This more general conjecture is suggested by dimensional reduction and by the cosmological billiards [16]. Given a theory consisting of gravity coupled to a dilaton and a set of forms it will, upon dimensional reduction on a torus, lead to a theory containing scalar fields as well as gauge fields. In general the scalars will not belong to a non-linear realisation. However, if one starts with a specific set of forms and dilaton coupling to the forms, one finds that the scalar fields do belong to a non-linear realisation. When this occurs, and the theory is reduced to three dimensions, all the fields can be reduced to scalars which belong to a coset space $\mathcal{G}/\mathcal{H}$ where $\mathcal{H}$ is the maximal compact subgroup of $\mathcal{G}$. In this procedure some of the forms may remain after dimensional reduction, but these may be swapped for scalars by dualisation. In fact one can find all of the finite-dimensional simple Lie algebras in this way [17]. It has become customary to refer to the unreduced theory from which one starts as the oxidation of the theory in three dimensions. The most important example is the 11-dimensional supergravity which can be thought of as a maximal oxidation of the three dimensional theory which has 128 scalar fields belonging to the $E_8/\,\mathrm{SO}(16)$ coset [18]. The oxidation of the three dimensional theory that has coset symmetry D_n is a theory in $n+2$ dimensions that consists of gravity coupled to a scalar field and a three-form field strength, the couplings of the fields being fixed. Reference [13] considered the dimensional reduction of certain theories of gravity coupled to a dilaton and certain n-forms and found that only for very specific couplings did the scalars belong to cosets and in all cases these couplings were one of the oxidised theories. The work on cosmological billiards referred to above was also extended to the oxidised theory for any $\mathcal{G}$ and it was found that the one dimensional motion took place in the Weyl chamber of the overextended algebras $\mathcal{G}^{++}$ [19].

The main results in this paper are as follows. Firstly, we have discovered sets of solutions of Einstein's equations coupled to scalar fields which form linear representations of the Weyl and outer automorphism group of $\mathcal{G}^{+++}$ for *any* $\mathcal{G}$ and secondly, we have shown that T and S-dualities in superstrings and in the bosonic string are encoded in suitable algebras $\mathcal{G}^{+++}$ in a way which depends specifically on the enlargement of $\mathcal{G}$ up to their triple Kac-Moody extension.

The content of the paper is organised as follows. In section 2, we lay out the general formalism used to uncover and test $\mathcal{G}^{+++}$ symmetries. We consider the Cartan subalgebra of $\mathcal{G}^{+++}$ with space-time fields as parameters. We identify the latter with the diagonal components of the metric and possible scalar dilaton-like fields. We find the invariant metric in the space of the fields and the effect of the Weyl transformations of $\mathcal{G}^{+++}$ on these fields. It is important to realise that although the Cartan subalgebra only contains commuting elements, it allows one to probe significant aspects of the $\mathcal{G}^{+++}$ algebra. One can think of this calculation as carrying out the full non-linear realisation for $\mathcal{G}^{+++}$, but then setting to zero all the fields except those in the Cartan subalgebra. In section 3, we show how the properties of dimensional reduction of 'maximally oxidised' theories can be understood in terms of the general framework laid out in section 2. We first recall how reducing these theories to three dimensions yields the symmetry of the simple Lie groups $\mathcal{G}$. We then show how these symmetries can be embedded in an algebra $\mathcal{G}^{+++}$ by using both the metric of the compactification torus and of the non-compact dimensions as parameters of the algebra. In this way, we relate the invariant metrics and the Weyl transformations of $\mathcal{G}$ to those of $\mathcal{G}^{+++}$. We then formulate a general theory of such Weyl preserving embeddings, which is to be used in the next two sections. In section 4, we study solutions of the gravity sector of all maximally oxidised theories. We focus our attention on the Kasner solutions of gravity with dilatons and show how the set of these solutions can be enlarged to form representations of the group $S(\mathcal{G}^{+++})$ of Weyl transformations and outer automorphisms of $\mathcal{G}^{+++}$. The connection of these results with the cosmological billiards is explained. Section 5 discusses how the group $S(\mathcal{G}^{+++})$ encodes the T and S-dualities of strings for those oxidised theories which can be interpreted as low energy effective actions of M-theory or string theories. When $\mathcal{G}^{+++}$ is restricted to its subgroups acting in the compact dimensions, one recovers well-known results. The present approach however points to more general links between string theories and gravity and in particular reveals the precise relation between an S-duality of the bosonic string and the S-duality in the heterotic string. More importantly perhaps, it shows the full power of $\mathcal{G}^{+++}$ when the non-compact dimensions are taken into account. Namely the rescaling of the minkowskian metric due to the transformations of the

Planck length under string dualities is automatically taken care of by $\mathcal{G}^{+++}$, providing a specific signature of this algebra. The significance of these results is discussed in section 6.

We now give a systematic account of symmetries in supergravity and string theories form which two essential elements emerge. The first one is the appearance of non-linear realisations of group symmetries in supergravity theories and the second crucial element is the idea that symmetries, originally discovered in what turned out to be dimensionally reduced theories, are really symmetries of full uncompactified theories.

Eleven dimensional supergravity has no scalars, and apart from the dilaton in IIA and the two scalar fields in IIB, the supergravity theories in ten dimensions possess no scalars either. In lower dimensions however, many scalars appear and are found to occur in non-linear realisations, as a consequence of supersymmetry. In such constructions, the scalars belong to cosets. This feature was first observed in the four dimensional $N = 4$ supergravity theory which possesses an $\mathrm{SL}(2,\mathbb{R})$ symmetry with $\mathrm{U}(1)$ as local subgroup, i.e. the scalars belong to the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ [20]. One of the most intriguing discoveries in this context is that the maximal supergravity theory in four dimensions contains 70 scalars and has E_7 symmetry [21]. In fact, when 11-dimensional supergravity is dimensionally reduced on a k-torus to $11-k$ dimensions for $k = 1,\ldots,8$, the resulting scalars can be expressed as a non-linear realisation [22] of the sequence of the finite-dimensional Lie algebras E_k. The role of $\mathrm{GL}(D)$ symmetries and dualities and their relationship with the Dynkin diagram of E_k was discussed in [23]. Dimensional reduction of 11-dimensional supergravity to three dimensions leads to a theory which is invariant under E_8 [18], to two dimensions leads to a theory that is invariant under the affine extension of E_8 [24], which is also called E_9 (and denoted in this paper by $E_8{}^{+}$), while dimensional reduction to one dimension is thought to result in a theory that is invariant under the hyperbolic algebra E_{10} (labelled hereafter as the overextension of E_8 and denoted by $E_8{}^{++}$) [25]. In fact the two scalars in the IIB supergravity theory belong to a $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ non-linear realisation [4] and as the dimensional reduction of this theory on a k-torus agrees with that of the IIA supergravity theory, one finds the same set of cosets as for this latter theory.

For many years it has been known that any string theory reduced on a torus is invariant under T-duality [26]. Indeed, it has been shown [27] that this is a symmetry to all orders of perturbation theory. It has also been conjectured that some string theories possess non-perturbative symmetries called S-dualities. The first such conjecture was about an $\mathrm{SL}(2,\mathbb{Z})$ symmetry in the context of the heterotic string reduced on a 6-torus [28]. It is believed that, when restricted to be defined on the integers, all the coset symmetries based on the finite-dimensional semi-simple Lie algebras discussed above are symmetries of string theory, called U-dualities [29]. It has also been shown [30, 31, 32] that the T-duality transformations for the IIA string theory in ten dimensions reduced on an k-torus for $k = 1,\ldots,10$ have a natural action on the moduli space of the k-torus that is the Weyl group of E_k. It has also been suggested that the closed bosonic string reduced on the torus associated with the unique self-dual twenty-six dimensional lorentzian lattice is invariant under the Borcherds fake monster algebra [33] and there is some evidence of Kac-Moody or possibly Borcherds structures in threshold corrections of the heterotic string reduced on a six dimensional torus [34].

Until recently it was not thought that the exceptional symmetries found in the dimensional reduction of 11-dimensional supergravity were symmetries of 11-dimensional supergravity itself. Some time ago however, it was shown [35] that 11-dimensional supergravity does possess an $\mathrm{SO}(1,2)\times\mathrm{SO}(16)$ symmetry, although the $\mathrm{SO}(1,10)$ tangent space symmetry is no longer apparent in this formulation. It has also been noticed that some of the objects associated with the exceptional groups emerging from the reductions appear naturally in the unreduced theory [36].

The coset construction for the scalars was extended [37] to include the gauge fields of supergravity theories. This method used generators that were inert under Lorentz transformations and, as such, it is difficult to extend the method further to include either gravity or the fermions. However, this construction did include the gauge and scalar fields as well as their duals, and as a consequence, the equations of motion for these fields could be expressed as a generalised self-duality condition.

More recently, it has been found that the dynamics of theories of gravity coupled to a dilaton and n-forms near a space-like singularity becomes a one dimensional motion with scattering taking place in the Weyl chambers of certain overextended Lie algebras [16, 38]. This motion is sometimes referred to as cosmological billiards. Supergravity in ten and eleven dimensions are such theories. One finds that 11-dimensional supergravity, IIA and IIB supergravities all lead to the same Weyl chamber which is that of overextended ${E_8}^{++}$, while type-I and heterotic supergravities lead to the Weyl chamber of B_8^{++} [38]. For gravity alone in D dimensions, the corresponding Weyl chamber is that of overextended A_{D-2}^{++} [38, 39].

Despite these observations it has been widely thought that the exceptional groups found in the dimensional reductions of the maximal supergravity theories can not be symmetries of 11-dimensional supergravity and must arise as a consequence of the dimensional reduction procedure. This is perhaps understandable given that the non-linear realisations or coset symmetries were associated with scalar fields and there are no scalar fields in 11-dimensional supergravity and the two scalars in the IIB theory only belong to the non-linear realisation based on $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$. However, it has been conjectured [12] that the 11-dimensional supergravity theory possesses a hidden E_{11} symmetry. This conjecture is based essentially on two results. Firstly, it had previously been shown [11] that the whole bosonic sector of 11-dimensional supergravity, including its gravitational sector, was a non-linear realisation. This placed the other bosonic fields of the theory on an equal footing with scalars and showed that non-linear realisation of symmetries can take place, whether or not there are scalars in the theory. Secondly, this construction contained substantial fragments of larger symmetries including A_{10} and the Borel subgroup of E_7 [12]. This, and other features of the non-linear realisation, suggested that these algebras should be incorporated into a Kac-Moody algebra, or in a more general symmetry. Although it was not proven that such a symmetry was realised, it was shown that if the symmetry of the theory included a Kac-Moody algebra then it must contain the very extended algebra of E_8, called E_{11} [12]. In this paper we shall write this triple extension of E_8 as E_8^{+++}. It was also shown that this construction could be generalised to the ten dimensional IIA [12] and IIB [40] supergravity theories and the corresponding Kac-Moody algebra was also E_8^{+++} in each case.

These ideas were taken up in reference [41] which considered 11-dimensional supergravity as a non-linear realisation of the E_{10} subalgebra of E_{11} in the small tension limit which played a crucial role in the work of references [38, 39]. These authors also introduced a new concept of level in the context of E_{10} which allowed them to deduce the representation content of E_{10} in terms of representations of A_9 at low levels. They showed that the 11-dimensional supergravity equations were E_{10}-invariant, up to level three, in the small tension limit, provided one adopted a particular map relating the fields at a given spatial point to quantities dependent only on time. In this limit the spatial dependence of the fields was very restricted. The U-duality groups has been enlarged into some super Borcherds algebras [42].

The approach given in reference [12] for M-theory was also applied to the closed bosonic string in twenty-six dimensions. Although this theory has no supersymmetry it is essentially unique with a corresponding low energy effective action. It was shown that this action, generalised to any dimension [11], was a non-linear realisation and on similar grounds it was argued [12] that in twenty-six dimensions it might possess a symmetry based on K_{27} or very extended algebra of D_{24}, denoted in this paper as D_{24}^{+++}. It was suggested that it was this symmetry that is responsible for the uniqueness and spectacular properties of this theory. This is consistent with the old idea [43] that the superstrings in ten dimensions are contained in the closed bosonic string. More recently this work has been extended to include the branes of the two theories [44]. In a similar spirit it has been conjectured [13] that gravity in D dimensions may possess a Kac-Moody symmetry which is A_{D-3}^{+++}.

2. Group theory of abelian configurations

As explained in the introduction there are indications [12] pointing towards the existence of a symmetry which could be E_8^{+++} in the M-theory approach to a unified theory of gravity and matter. Similarly, the symmetry D_{24}^{+++} is thought to underlie the bosonic string theory [12]. It has also been suggested that A_{D-3}^{+++} could be a symmetry of gravity [13]. Sections 4 and 5 will provide explicit examples which not only corroborate these conjectures but, remarkably, provide evidence for the physical relevance of the triple extension $\mathcal{G}^{+++}$ of any simple Lie algebra $\mathcal{G}$.

In this section, we associate fields to the algebra $\mathcal{G}^{+++}$ in order to test these symmetries. To avoid the complexity of a full non-linear realisation of the algebra, we restrict our attention to its Cartan subalgebra. We will see that this corresponds to consider the diagonal components of the metric field and possibly scalar 'dilaton' fields. We then calculate the effect of the Weyl transformations and outer automorphism of $\mathcal{G}^{+++}$ on these space-time fields. This provides a tool for testing hidden symmetries in subsequent sections.

2.1 General formalism and the preferred subgroup

Any finite-dimensional simple Lie algebra $\mathcal{G}$ can be extended in a unique way to a Kac-Moody algebra $\mathcal{G}^{+++}$. This procedure increases the rank of the algebra by three and we refer the reader to reference [15] for a discussion of this process. The Dynkin diagrams of all the simple algebras $\mathcal{G}^{+++}$ are depicted in figure 1 and 2.

JHEP09(2003)020

Let us denote the rank of $\mathcal{G}^{+++}$ by r and let E_m, F_m and H_m, $m = 1, 2, \dots, r$, be its Chevalley generators satisfying

$$\begin{aligned}
[H_m, H_n] &= 0\,, & [H_m, E_n] &= A_{mn} E_n\,, \\
[H_m, F_n] &= -A_{mn} F_n\,, & [E_m, F_n] &= \delta_{mn} H_m\,,
\end{aligned} \tag{2.1}$$

where A_{mn} is the Cartan matrix. The Cartan subalgebra is generated by H_m, while the positive (negative) generators are the commutators of the E_m (F_n) subject to the Serre relations

$$[E_m, [E_m, \dots, [E_m, E_n]]] = 0\,, \qquad [F_m, [F_m, \dots, [F_m, F_n]]] = 0\,, \tag{2.2}$$

where the number of E_m (F_m) acting on E_n (F_n) is given by $1 - A_{mn}$. There exists a scalar product $\langle\,,\rangle$ on an r-dimensional vector space such that the Cartan matrix can be expressed in terms of the simple roots as

$$A_{mn} = 2\frac{\langle \alpha_m, \alpha_n \rangle}{\langle \alpha_m, \alpha_m \rangle}\,. \tag{2.3}$$

Any given algebra $\mathcal{G}^{+++}$ contains a preferred subalgebra[1] GL(D) with $D \leq r$ where D will be identified with the number of space-time dimensions through the introduction of metric fields. The dimension D is thus encoded in $\mathcal{G}^{+++}$, as will now be explained.

The generators of GL(D) are taken to be $K^a{}_b$, $a, b = 1, 2, \dots, D$ with commutation relations

$$[K^a{}_b, K^c{}_d] = \delta^c{}_b K^a{}_d - \delta^a{}_d K^c{}_b\,. \tag{2.4}$$

The corresponding group elements are

$$g = e^{h^b{}_a(x^\mu) K^a{}_b}\,, \tag{2.5}$$

where the x^μ are coordinates on a D-dimensional manifold. We shall express the group parameters $h^b{}_a(x^\mu)$ in terms of the metric tensor $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu = \vec{e}_a.\vec{e}_b\, e^a{}_\mu e^b{}_\nu$ at the point x^μ. The vectors $\vec{e}_a$ form an orthonormal basis in the tangent plane at the point x^μ. The group generators $K^a{}_b$ act on any vector $\vec{z}$ in the plane according to

$$K^a{}_b = z^a \frac{\partial}{\partial z^b}\,. \tag{2.6}$$

Thus, the group element eq. (2.5) generates the linear transformation

$$z'^c = g\, z^c = e^{h^b{}_a(x^\mu) z^a \partial/\partial z^b} z^c = \left(e^{h(x^\mu)}\right)^c{}_a z^a\,, \tag{2.7}$$

where h is the matrix whose elements are the fields $h^b{}_a$. Transforming the basis $\vec{e}_a$ to $\vec{e}_\mu = e^a{}_\mu \vec{e}_a$ one gets

$$e^a{}_\mu(x^\mu) = \left(e^{h(x^\mu)}\right)^a{}_\mu\,. \tag{2.8}$$

The vielbein group SO(D) makes it evident that the metric fields $g_{\mu\nu}(x)$ live in the coset GL(D)/ SO($1, D-1$).

[1]Throughout the paper we use the same notation for groups and algebras.

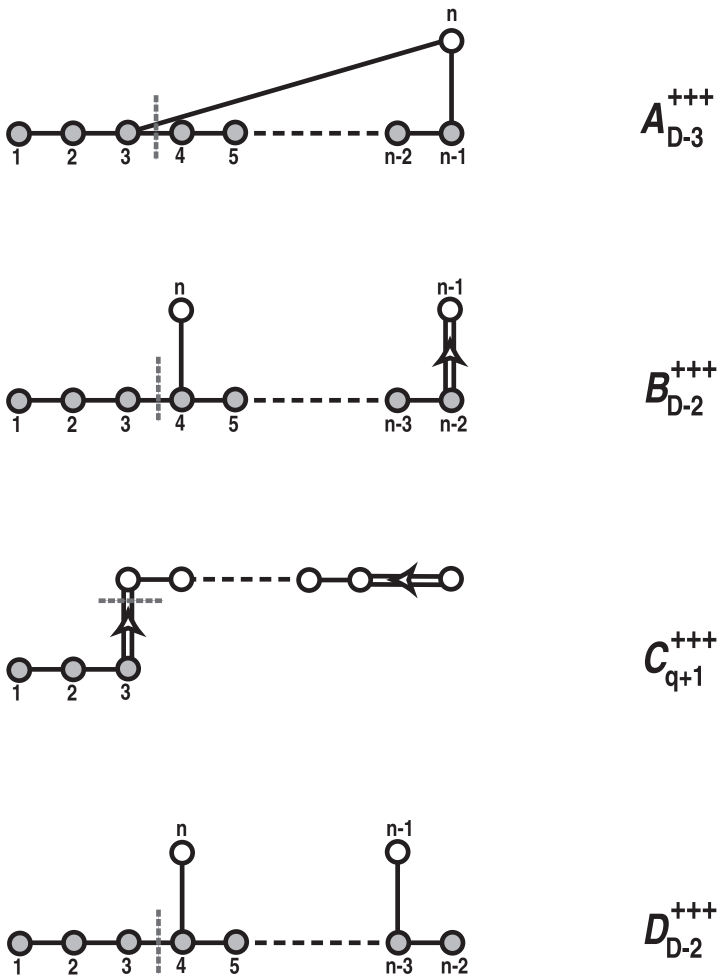

Figure 1: Dynkin diagram of $\mathcal{G}^{+++}$: A, B, C, D series. The nodes of the gravity line are shaded. The nodes labelled 1,2,3 are the Kac-Moody extensions of the Lie algebras.

A more elaborate derivation of eq. (2.8) yielding a covariant coupling of the metric to other fields is obtained using a non-linear realisation of GL(D) with local subgroup SO($1, D-1$). One first introduces the extension IGL(D) of the group GL(D) by adding D momentum generators P_a to its generators $K^a{}_b$ and carries out simultaneously the non-linear realisation with the conformal group [45, 11].

The selection of the preferred subalgebra GL(D) of $\mathcal{G}^{+++}$ is achieved as follows. Starting from the root of the Dynkin diagram that extends $\mathcal{G}$ up to $\mathcal{G}^{+++}$, labelled 1 in figure 1 and figure 2, one follows the line of long roots up to the last one whose Chevalley generator has the form $H_m = \delta^a_m (K^a{}_a - K^{a+1}{}_{a+1})$. This line constitutes the Dynkin diagram of an A_{D-1} subgroup and is referred to as the gravity line.[2]

The above expressions for the H_m generators are identified by noting that the simple positive roots of SL(D) are $K^a{}_{a+1}$, $a = 1, \ldots, D-1$. The additional commuting generator

[2]In case there is a bifurcation in the Dynkin diagram, we select the line consistent with the GL($D-3$) subgroup of theory in three dimensions that results from the corresponding maximally oxidised theory.

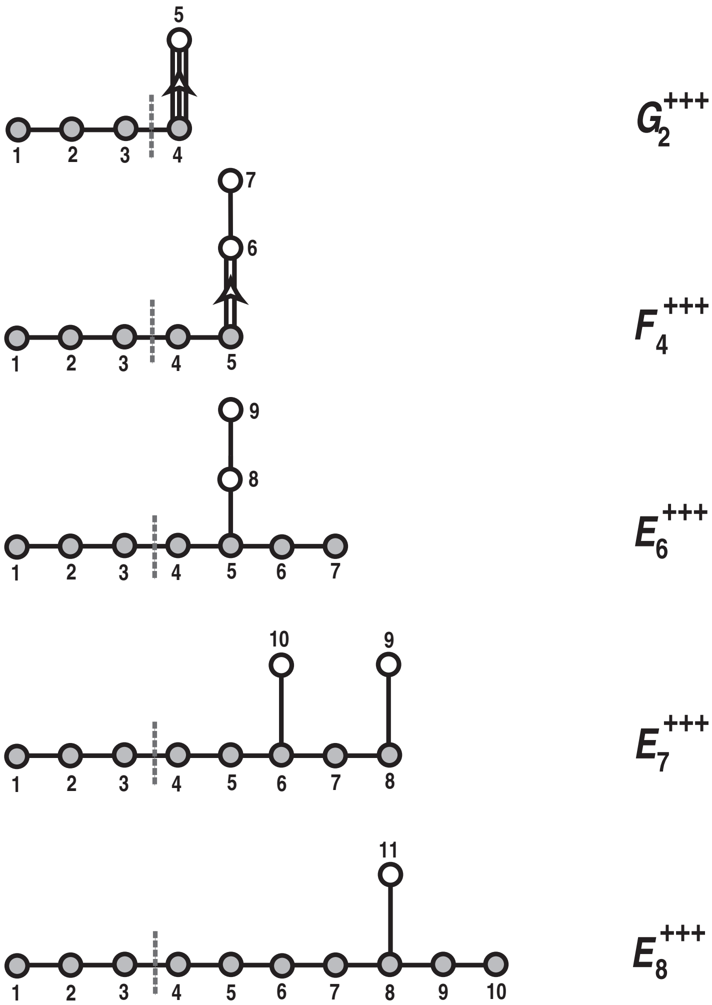

Figure 2: Dynkin diagram of $\mathcal{G}^{+++}$: exceptional algebras.

$\sum_{a=1}^{D} K^a{}_a$ extends SL(D) to the preferred GL(D). The rank r of some of the algebras $\mathcal{G}^{+++}$ is equal to the rank D of the preferred GL(D) subalgebra. In these cases the embedding of GL(D) is such that the Cartan generators H_m of $\mathcal{G}^{+++}$ in the Chevalley basis can be expressed in terms of the generators $K^a{}_a, a = 1, \ldots, D$ and vice versa. For some $\mathcal{G}^{+++}$ algebras however, $D < r$ and we enlarge the preferred subalgebra to

$$\mathrm{GL}(D) \oplus R_1 \oplus \cdots \oplus R_q\,, \tag{2.9}$$

where $[R_u, K^a{}_b] = 0$, $u = 1, \ldots, q$. The number of extra commuting generators R_u is $q = r - D$. These additional generators will be associated with scalar 'dilaton' fields Φ^u. The occurrence of such fields will be corroborated by the analysis of section 3. For each simple $\mathcal{G}^{+++}$, the number q of dilatons can be read off its Dynkin diagram and is given in table 1.

We now restrict ourselves to the Cartan subalgebra of $\mathcal{G}^{+++}$ which we express in terms of the generators associated with gravity and scalar dilaton fields, namely we consider the

JHEP09(2003)020

L-basis defined by

$$L_i = \{K^a{}_a, R_u\} \qquad i = 1, 2, \ldots, q + D = r\,. \tag{2.10}$$

The corresponding abelian group element g is

$$g = \exp\left(\sum_{a=1}^{D} p^a K^a{}_a\right) \exp\left(\sum_{u=1}^{q} \Phi^u R_u\right). \tag{2.11}$$

Comparing eq. (2.11) with the group element given in eq. (2.5) and using eq. (2.8), we find that taking a group element restricted to the Cartan subalgebra corresponds in the gravity sector to taking a vielbein that is diagonal. The coordinates p^a are thus related to a diagonal metric configuration by[3]

$$e^a{}_\mu = \left(e^h\right)^a{}_\mu = e^{p^a}\,\delta^a_\mu \qquad \text{or} \qquad g_{aa} = e^{2p^a}\eta_{aa}\,. \tag{2.12}$$

As already mentioned, the additional $r - D$ coordinates are the dilaton fields. The group element eq. (2.11) can also be written in the Chevalley basis as

q	D	$\mathcal{G}^{+++}$
0	D	A^{+++}_{D-3}
1	D	B^{+++}_{D-2} and D^{+++}_{D-2}
q	4	C^{+++}_{q+1}
1	8	E^{+++}_6
1	9	E^{+++}_7
0	11	E^{+++}_8
0	5	G^{+++}_2
1	6	F^{+++}_4

Table 1: $\mathrm{GL}(D) \oplus R_1 \oplus \cdots \oplus R_q$ subalgebra of $\mathcal{G}^{+++}$.

$$g = e^{q^m H_m}\,, \tag{2.13}$$

and the H_m are related to the L_i by a linear relation

$$H_m = r_m^{\ i}\, L_i\,. \tag{2.14}$$

Given any symmetrisable Kac-Moody algebra there exists, up to a numerical factor, a unique scalar product defined on the algebra that is invariant under the adjoint action of the algebra [46]. For a finite-dimensional simple Lie algebra this is just the Killing form which can be expressed as the trace of the generators in any finite-dimensional representation. While the expression for the scalar product on the full Kac-Moody algebra may be complicated, its expression on the Cartan subalgebra, up to a numerical scale factor, is given in terms of the Cartan matrix by [46]

$$(G_c)_{mn} = \frac{2A_{mn}}{\langle\alpha_n, \alpha_n\rangle} = \frac{4\langle\alpha_m, \alpha_n\rangle}{\langle\alpha_m, \alpha_m\rangle\langle\alpha_n, \alpha_n\rangle}\,. \tag{2.15}$$

This invariant metric will be used in deriving Weyl and other symmetries in theories containing gravity and extra degrees of freedom. We now show how to obtain the invariant metric in the space of the physical fields defined in the L-basis.

We have

$$g = e^{q^m H_m} = e^{p^i L_i} \qquad \text{where} \qquad p^i = q^m r_m^{\ i} \qquad \text{or} \qquad \boldsymbol{p^T} = \boldsymbol{q^T} \cdot \boldsymbol{r}\,, \tag{2.16}$$

and the p^i are

$$p^i = \{p^a, \Phi^u\}\,, \tag{2.17}$$

[3]The indices in g_{aa} are identified by the correspondence between p^a and $e^a{}_\mu$ in a given frame.

with p^a given in eq. (2.12). In terms of the L-basis the invariant metric is

$$G^{+++}_{ij} = \left[\boldsymbol{r^{-1}} \cdot \boldsymbol{G_c} \cdot (\boldsymbol{r^{-1}})^{\boldsymbol{T}}\right]_{ij} , \tag{2.18}$$

where $\boldsymbol{G_c}$ is given in eq. (2.15).

We shall see in the examples below and in section 3 that G^{+++}_{ij} has the same form for *all* the algebras $\mathcal{G}^{+++}$ and is equal to

$$\boldsymbol{G}^{+++} = \frac{1}{2}\boldsymbol{I}_{r-D} \oplus \left(\boldsymbol{I}_D - \frac{1}{2}\boldsymbol{\Xi}_D\right), \tag{2.19}$$

where $\boldsymbol{\Xi}_D$ is a D-dimensional matrix with all entries equal to one. The reason behind this universality is that eq. (2.18) can be formally written as

$$G^{+++}_{ij} = (\boldsymbol{r^{-1}})_i{}^m \langle H_m, H_n\rangle (\boldsymbol{r^{-1}})_j{}^n = \langle L_i, L_j\rangle , \tag{2.20}$$

and that the last expression involves only generators of the preferred subgroup. The part of the scalar product involving the generators of $\mathrm{SL}(D)$ is fixed as it must be the unique, up to a scale, invariant scalar product on this algebra. We have used the arbitrariness of the normalisation of the dilaton fields to fix the factor multiplying $\boldsymbol{I}_{r-D}$ in the metric eq. (2.19) to be $1/2$. Note that we can also write eq. (2.18) as

$$(\boldsymbol{r})_m{}^i G^{+++}_{ij} (\boldsymbol{r})_n{}^j = (G_c)_{mn} = \langle \alpha^\vee_m, \alpha^\vee_n\rangle , \tag{2.21}$$

where the coroots $\alpha^\vee_n$ are

$$\alpha^\vee_n = \frac{2\alpha_n}{\langle \alpha_n, \alpha_n\rangle} . \tag{2.22}$$

Eq. (2.21) identifies the matrix of the contravariant components of the coroots in the L-basis with the matrix $\boldsymbol{r}$.

We now show how to use our formalism to obtain the action of the $\mathcal{G}^{+++}$ Weyl transformations on the fields.

The Weyl transformations S_α of any Kac-Moody algebra are reflections in the planes perpendicular to the real roots α of the algebra. Their action on an arbitrary weight ω is given by

$$\omega' \stackrel{\text{def}}{=} S_\alpha\, \omega = \omega - 2\frac{\langle \omega, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha . \tag{2.23}$$

The Weyl group is generated by the reflections associated with the simple roots α_n. Their action on the simple coroots $\alpha^\vee_m$ is

$$\alpha^{\vee\prime}_m \stackrel{\text{def}}{=} S_{\alpha_n} \alpha^\vee_m = (s_{\alpha_n})_m{}^p \alpha^\vee_p = \alpha^\vee_m - A_{mn}\, \alpha^\vee_n . \tag{2.24}$$

Since they are reflections, $\boldsymbol{s}^2_{\alpha_a} = \boldsymbol{1}$. Although the full Kac-Moody algebra does not carry a representation of the Weyl group, the Cartan subalgebra does, and it takes the form

$$H'_m \stackrel{\text{def}}{=} S_{\alpha_n} H_m = (s_{\alpha_n})_m{}^p H_p . \tag{2.25}$$

The corresponding action on the fields q^m in the group element eq. (2.13) is given by

$$\boldsymbol{q'^T} = \boldsymbol{q^T} \cdot \boldsymbol{s}_{\alpha_n} . \tag{2.26}$$

The invariant scalar product on any Kac-Moody algebra, discussed above, is also invariant under Weyl reflections. When restricted to the Cartan subalgebra elements this can be seen as consequence of the fact that reflections preserve the scalar products of the roots. Consequently the metric $(G_c)_{mn}$ of eq. (2.15) is invariant under the Weyl group, i.e. $\boldsymbol{s}_{\alpha_n} \cdot \boldsymbol{G_c} \cdot \boldsymbol{s}^{\boldsymbol{T}}_{\alpha_n} = \boldsymbol{G_c}$ for all simple roots α_n. Using the relation eq. (2.16) we find that the generators of the Weyl group acting on the physical fields p^i are given by

$$\boldsymbol{p}'^{\boldsymbol{T}} = \boldsymbol{p}^{\boldsymbol{T}} \cdot \boldsymbol{t}_{\alpha_n} \,, \tag{2.27}$$

where

$$\boldsymbol{t}_{\alpha_n} = \boldsymbol{r}^{\boldsymbol{-1}} \cdot \boldsymbol{s}_{\alpha_n} \cdot \boldsymbol{r} \,. \tag{2.28}$$

The corresponding action of the Weyl group on the generators L_i is thus given by $L_i' = (t_{\alpha_n})_i{}^j L_j$ and the invariance of the metric G_{ij} is expressed by $\boldsymbol{t}_{\alpha_n} \cdot \boldsymbol{G} \cdot \boldsymbol{t}^{\boldsymbol{T}}_{\alpha_n} = \boldsymbol{G}$. Since we have identified diagonal space-time metrics in eq. (2.12) in terms of p^a, we can read off the effect of a Weyl transformation on such metrics. For the algebras that require additional R_u generators we can extend this procedure to Weyl induced transformations on the diagonal metric components and on the fields Φ^u. This is one of the central results of this section and is worked out in detail for several important cases below.

The above construction can be viewed as a restriction of a non-linear realisation of $\mathcal{G}^{+++}$. To see this, let us consider a non-linear realisation of $\mathcal{G}^{+++}$ where the local subgroup is the subgroup invariant under the Cartan involution. The generators of the Cartan subalgebra of $\mathcal{G}^{+++}$ are not contained in the local subgroup as they are not invariant under the Cartan involution. As such, they lead to the finite set of fields q^m in the non-linear realisation. We can think of the group elements eq. (2.13) as those of the full non-linear realisation, but with all the fields set to zero except for those associated with the Cartan subalgebra. It is important to realise that in doing so we can capture more information than that just contained in the Cartan subalgebra. This occurs in two ways: we have used the information about the embedding of the physically relevant preferred $\mathrm{GL}(D) \oplus R_1 \oplus \cdots \oplus R_q$ in the full $\mathcal{G}^{+++}$ algebra and we have also used a scalar product, or metric $(G_c)_{mn}$, which is the restriction of the scalar product invariant under the full $\mathcal{G}^{+++}$ algebra. This explains why we shall find results relevant to M and more general theories from an analysis of their gravity sector.

We now illustrate the above general formalism by applying it to several important examples.

2.2 E_8^{+++} or M-theory

We now consider the algebra E_8^{+++} which is the suggested symmetry of M-theory [12]. The algebra E_8^{+++} admits $\mathrm{SL}(11) = A_{10}$ as its gravity line as can be seen on the Dynkin diagram of figure 2: it is obtained by deleting the node labelled n. Since the rank of E_8^{+++} is eleven, which is also the number of commuting generators $K^a{}_a$ in GL(11), there are no additional R_u generators. The relation between the commuting generators $K^a{}_a$ of the preferred $\mathrm{GL}(D)$ and the Cartan generators H_m of E_8^{+++} in the Chevalley basis follows from comparing the commutation relations eqs. (2.1) and (2.4) and from the identification of the simple roots of E_8^{+++}. These are $E_m = \delta_m^a K^a{}_{a+1}$, $m = 1, \ldots, 10$ and $E_{11} = R^{9\,10\,11}$ where R^{abc} is a generator in E_8^{+++} that is a third rank anti-symmetric tensor under SL(11) and thus obeys the equation [12]

$$[K^a{}_b, R^{c_1c_2c_3}] = \delta_b^{c_1} R^{ac_2c_3} + \cdots \,, \tag{2.29}$$

where $+\cdots$ denotes the corresponding anti-symmetrisation. This also follows from the analysis of section 3. Given this commutator, it is straightforward to verify that the following relations between H_m and $K^a{}_a$ are correct[4] as they reproduce the Cartan matrix A_{mn} of E_8^{+++} using the relation $[H_m, E_n] = A_{mn}E_n$.

$$H_m = \delta_m^a (K^a{}_a - K^{a+1}{}_{a+1}) \qquad m = 1, \ldots, 10 \tag{2.30}$$

$$H_{11} = -\frac{1}{3}(K^1{}_1 + \cdots + K^8{}_8) + \frac{2}{3}(K^9{}_9 + K^{10}{}_{10} + K^{11}{}_{11})\,. \tag{2.31}$$

Since $H_m = r_m{}^a K^a{}_a$ one readily reads off the matrix $\boldsymbol{r}$. The metric G_{ij} in the L-basis can therefore be computed using eq. (2.18). One gets

$$\boldsymbol{G}^{+++} = \boldsymbol{I}_{11} - \frac{1}{2}\boldsymbol{\Xi}_{11} \text{ and its inverse } \boldsymbol{G}^{-1}_{+++} = \boldsymbol{I}_{11} - \frac{1}{9}\boldsymbol{\Xi}_{11}\,. \tag{2.32}$$

In calculating the inverse we use the fact that for any D the inverse of the matrix $\boldsymbol{I}_D + a\boldsymbol{\Xi}_D$ is $\boldsymbol{I}_D - [a/(1+Da)]\boldsymbol{\Xi}_D$.

We now give the effect of the Weyl transformations on the p^a fields. This is computed via eq. (2.28), or equivalently, in the following way. Let us first consider the Weyl transformation generated by a simple root α_n, $n \neq 1, 8, 10, 11$. Its action on the Chevalley generators H_m is easily read off eqs. (2.24), (2.25):

$$\begin{aligned} S_{\alpha_n} H_m &= H_m \qquad m \neq n\,, \quad m \neq n \pm 1\,, \\ S_{\alpha_n} H_n &= -H_n \qquad S_{\alpha_n} H_{n\pm 1} = H_n + H_{n\pm 1}\,. \end{aligned} \tag{2.33}$$

It is then straightforward to see from eq. (2.30) that the corresponding transformation on $K^a{}_a$ for $a = n$ is given by

$$K^a{}_a \leftrightarrow K^{a+1}{}_{a+1}\,. \tag{2.34}$$

This relation is actually valid for $n = 1, \ldots 10$, as is easily verified separately for $n = 1, 8, 10$. Hence on the fields $p^a (g_{aa} = e^{2p^a}\eta_{aa})$, the Weyl transformation associated to a simple root α_n, $n = 1, \ldots, 10$ induces the transformation

$$p^a \leftrightarrow p^{a+1} \qquad a = n\,, \quad n \neq 11\,, \tag{2.35}$$

all other p^a being unchanged. In other words each such Weyl transformation just interchanges two of the diagonal components of the metric.

It remains to find the action of the Weyl transformation generated by the simple root α_{11}. In this case

$$\begin{aligned} S_{\alpha_{11}} H_m &= H_m \qquad m \neq 8\,, \qquad m \neq 11\,, \\ S_{\alpha_{11}} H_{11} &= -H_{11} \qquad S_{\alpha_{11}} H_8 = H_8 + H_{11}\,. \end{aligned} \tag{2.36}$$

[4] In what follows, the indices in the diagonal $K^a{}_a$ are considered as a single index as far as summation conventions are concerned.

Examining the H_m that are unchanged we conclude that

$$K^a{}_a \to K^a{}_a + y \quad a = 1, \ldots, 8 \qquad K^a{}_a \to K^a{}_a + x \quad a = 9, 10, 11\,. \tag{2.37}$$

It is then straightforward to show that $y = 0$ and $x = -H_{11}$. As a result, we find that

$$S_{\alpha_{11}} K^a{}_a = K^a{}_a\,, \qquad a = 1, \ldots, 8 \tag{2.38}$$
$$S_{\alpha_{11}} K^a{}_a = K^a{}_a + \frac{1}{3}(K^1{}_1 + K^2{}_2 + \cdots + K^8{}_8) - \frac{2}{3}(K^9{}_9 + K^{10}{}_{10} + K^{11}{}_{11}) \qquad a = 9, 10, 11\,.$$

Hence we conclude that

$$p'^a = p^a + \frac{1}{3}(p^9 + p^{10} + p^{11}) \qquad a = 1, \ldots, 8 \tag{2.39}$$
$$p'^a = p^a - \frac{2}{3}(p^9 + p^{10} + p^{11}) \qquad a = 9, 10, 11\,. \tag{2.40}$$

These relations will be used in sections 4 and 5 in the context of M-theory whose effective bosonic lagrangian is [8]

$$\mathcal{L}^{(11)} = \sqrt{-g^{(11)}}\left(R^{(11)} - \frac{1}{2.4!}F^{(3)}_{\mu\nu\sigma\tau}F^{(3)\mu\nu\sigma\tau} + CS\text{-term}\right). \tag{2.41}$$

One can generalise the above discussion for the $E^{(k)}$, $k \le 11$ subgroups obtained by putting $q_1, \ldots, q_{11-k} = 0$. All the above discussion and formulae hold after making the appropriate changes in the labelling of indices. The only exception is the metric eq. (2.32) which is given by, (see also reference [32])

$$\boldsymbol{G^{(k)}} = \boldsymbol{I}_k + \frac{1}{9-k}\boldsymbol{\Xi}_k \text{ and its inverse by } \boldsymbol{G^{-1}_{(k)}} = \boldsymbol{I}_k - \frac{1}{9}\boldsymbol{\Xi}_k\,. \tag{2.42}$$

We shall come back to this point, and in particular to the case $k = 9$, in section 3.

2.3 D^{+++}_{D-2} and the bosonic string

The closed bosonic string in twenty-six dimensions is thought to possess a D^{+++}_{24} symmetry [12]. The Dynkin diagram of D^{+++}_{D-2} is given in figure 1. The gravity line $\mathrm{SL}(D) = A_{D-1}$ is obtained by deleting the two nodes n $(= D+1)$ and $n-1$ in the Dynkin diagram. The rank of D^{+++}_{D-2} is $D+1$ and we must add one generator R associated with the dilaton, as indicated in table 1.

The simple roots of D^{+++}_{D-2} are $E_m = \delta^a_m K^a{}_{a+1}$, $m = 1, \ldots, D-1$, $E_{n-1} = R^{n-2\,n-1}$ and $E_n = R^{5\ldots n-1}$ where R^{ab} and $R^{a_1\ldots a_{n-5}}$ are generators of D^{+++}_{D-2} that are 2 and $n-5$ rank anti-symmetric tensors under $\mathrm{SL}(D)$ and thus obey the equations [12]

$$[K^a{}_b, R^{c_1c_2}] = \delta^{c_1}_b R^{ac_2} + \cdots\,, \qquad [K^a{}_b, R^{c_1\ldots c_{n-5}}] = \delta^{c_1}_b R^{a\ldots c_{n-5}} + \cdots\,, \tag{2.43}$$

as well as

$$[R, R^{c_1c_2}] = \frac{l}{2}R^{c_1c_2}\,, \qquad [R, R^{c_1\ldots c_{n-5}}] = -\frac{l}{2}R^{c_1\ldots c_{n-5}}\,, \tag{2.44}$$

with

$$l = \left(\frac{8}{D-2}\right)^{1/2}. \tag{2.45}$$

These also follow from the analysis of section 3. The normalisation of the dilaton generator in eq. (2.45) is chosen for convenience. From these commutators, one gets as in the previous case the relations between the generators H_m and the generators $\{K^a{}_a, R\}$, namely

$$\begin{aligned} H_m &= \delta^a_m (K^a{}_a - K^{a+1}{}_{a+1}) \qquad m = 1, \dots, n-2 && (2.46)\\ H_{n-1} &= -\frac{2}{(D-2)}(K^1{}_1 + \dots + K^{n-3}{}_{n-3}) + \\ &+ \frac{(D-4)}{(D-2)}(K^{n-2}{}_{n-2} + K^{n-1}{}_{n-1}) + l\, R\,, && (2.47)\\ H_n &= -\frac{(D-4)}{(D-2)}(K^1{}_1 + \dots + K^4{}_4) + \\ &+ \frac{2}{(D-2)}(K^5{}_5 + \dots + K^{n-1}{}_{n-1}) - l\, R\,. && (2.48) \end{aligned}$$

The metric G_{ij} appropriate to the p^i fields can be calculated using eq. (2.18). One obtains

$$\boldsymbol{G}^{+++} = \frac{1}{2}\boldsymbol{I}_1 \oplus \left(\boldsymbol{I}_D - \frac{1}{2}\Xi_D\right), \qquad (2.49)$$

in agreement with the general expression eq. (2.19). Its inverse is given by

$$\boldsymbol{G}^{-1}_{+++} = 2\,\boldsymbol{I}_1 \oplus \left(\boldsymbol{I}_D - \frac{1}{D-2}\Xi_D\right). \qquad (2.50)$$

The determination of the Weyl transformations follows the same steps as in the previous example. The Weyl transformations corresponding to simple roots in the gravity line induce an exchange of neighbouring components of the diagonal metric and leave Φ invariant.

The Weyl transformation corresponding to the simple root α_{n-1} induces the changes

$$\begin{aligned} &K^a{}_a \to K^a{}_a \quad a = 1, \dots, n-3 \\ &K^a{}_a \to K^a{}_a - H_{n-1} \quad a = n-2, n-1 \qquad R \to R - \frac{l}{2}H_{n-1}\,, && (2.51) \end{aligned}$$

from which we derive

$$\begin{aligned} p'^a &= p^a + \frac{2}{(D-2)}(p^{n-2} + p^{n-1}) + \frac{l}{D-2}\Phi \qquad a = 1, \dots, n-3 && (2.52)\\ p'^a &= p^a - \frac{(D-4)}{(D-2)}(p^{n-2} + p^{n-1}) - \frac{l\,(D-4)}{2\,(D-2)}\Phi \qquad a = n-2, n-1 && (2.53)\\ \Phi' &= \Phi - l(p^{n-2} + p^{n-1}) - \frac{4}{D-2}\Phi\,. && (2.54) \end{aligned}$$

The Weyl transformation corresponding to the simple root α_n induces the changes

$$\begin{aligned} &K^a{}_a \to K^a{}_a \quad a = 1, \dots, 4 \\ &K^a{}_a \to K^a{}_a - H_n \quad a = 5, \dots, n-1 \qquad R \to R + \frac{l}{2}H_n\,. && (2.55) \end{aligned}$$

Hence p^a and Φ transform as

$$p'^a = p^a + \frac{D-4}{D-2}(p^5 + \cdots + p^{n-1}) - \frac{l\,(D-4)}{2\,(D-2)}\Phi \qquad a = 1, \ldots, 4 \tag{2.56}$$

$$p'^a = p^a - \frac{2}{D-2}(p^5 + \cdots + p^{n-1}) + \frac{l}{(D-2)}\Phi \qquad a = 5, \ldots, n-1 \tag{2.57}$$

$$\Phi' = \Phi + l\,(p^5 + \cdots + p^{n-1}) - \frac{4}{(D-2)}\Phi\,. \tag{2.58}$$

Unlike the previous case, D^{+++}_{D-2} possesses an outer automorphism which manifests itself in the symmetry of the Dynkin diagram found by exchanging nodes $n-2$ and $n-1$. This will also be an invariance of the metric and we can compute its effect on the fields in the same way as for Weyl transformations. Under the outer automorphism the H_m transform as

$$H'_{n-2} = H_{n-1} \qquad H'_{n-1} = H_{n-2} \qquad H'_m = H_m \quad m = 1, \ldots, n-3, n\,. \tag{2.59}$$

Hence

$$\begin{aligned} K'^{n-1}{}_{n-1} &= K^{n-1}{}_{n-1} + H_{n-2} - H_{n-1}\,, \qquad R' = R + \frac{l}{4}(H_{n-2} - H_{n-1})\,, \\ K'^a{}_a &= K^a{}_a \qquad a = 1, \ldots, n-2\,. \end{aligned} \tag{2.60}$$

The corresponding change induced on the p^a and Φ is given by

$$p'^a = p^a + \frac{2}{(D-2)}p^{n-1} + \frac{l}{2\,(D-2)}\Phi \qquad a = 1, \ldots, n-2 \tag{2.61}$$

$$p'^{n-1} = p^{n-1} - \frac{2(D-3)}{(D-2)}p^{n-1} - \frac{l\,(D-3)}{2\,(D-2)}\Phi \tag{2.62}$$

$$\Phi' = \Phi - \frac{2}{D-2}\Phi - l\,p^{n-1}\,. \tag{2.63}$$

These relations will be used in sections 4 and 5 in the context of the bosonic string theory whose effective lagrangian is, for $D = 26$, given by [47]

$$\mathcal{L}^D = \sqrt{-g^D}\left(R^D - \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2\cdot 3!}e^{-l\Phi}F^{(3)}_{\mu\nu\sigma}F^{(3)\mu\nu\sigma}\right), \tag{2.64}$$

where l is defined by eq. (2.45).

2.4 B^{+++}_{D-2} and the heterotic string

The Dynkin diagram of B^{+++}_{D-2} is given in figure 1. The gravity line $\mathrm{SL}(D) = A_{D-1}$ is obtained by deleting the two nodes $n\,(= D+1)$ and $n-1$ in the Dynkin diagram, where $n-1$ is the short root. The rank of B^{+++}_{D-2} is $D+1$ and we must again add one generator R associated with the dilaton, as indicated in table 1.

Following the previous method, one gets results similar to those for the D-series. The metric for the p^i fields is given by eqs. (2.49) and (2.50), in agreement with the universal formula eq. (2.19). Again, the Weyl transformations corresponding to simple roots in the gravity line induce an exchange of neighbouring components of the diagonal metric and leave Φ invariant. The Weyl transformation for the simple root α_n yields the same result as previously, namely eqs. (2.56)- (2.58) while the Weyl transformation generated by the short root α_{n-1} yields the transformation eqs. (2.61)–(2.63). Thus we see that the outer automorphism of the D-series is converted in the B-series to the Weyl reflection corresponding to the short root. These relations will be used in section 5 in the context of the heterotic string theory whose bosonic effective lagrangian is, for $D = 10$, related to [7]

$$\mathcal{L}^D = \sqrt{-g^D}\left(R^D - \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2.3!}e^{-l\Phi}F^{(3)}_{\mu\nu\sigma}F^{(3)\mu\nu\sigma} - \right.$$
$$\left. - \frac{1}{2.2!}e^{-(l/2)\Phi}F^{(2)}_{\mu\nu}F^{(2)\mu\nu} + C.S.\right), \tag{2.65}$$

where l is given by eq. (2.45).

2.5 A^{+++}_{D-3} or gravity

The Dynkin diagram of A^{+++}_{D-3} is given in figure 1. The gravity line $\mathrm{SL}(D) = A_{D-1}$ is obtained by deleting the node $n = D$ in the Dynkin diagram. The rank of A^{+++}_{D-3} is D and we need no further dilaton generator, as indicated in table 1.

The simple roots of A^{+++}_{D-3} are $E_m = \delta^a_m K^a{}_{a+1}$, $m = 1, \ldots, n-1$ and $E_n = R^{4\ldots n-1,n}$ where $R^{a_1\ldots a_{n-4},b}$ is totally anti-symmetric in its a indices and the part antisymmetrised in all its indices vanishes [13, 48]. This generator of A^{+++}_{D-3} transforms under $\mathrm{SL}(D)$ as the indices suggest and obeys the commutator

$$[K^a{}_b, R^{c_1\ldots c_{n-4},d}] = \delta^{c_1}_b R^{a\ldots c_{n-4},d} + \cdots + \delta^d_b R^{c_1\ldots c_{n-4},a}\,. \tag{2.66}$$

These also follow from the analysis of section 3. From these commutators, one gets the following relations

$$\begin{aligned} H_m &= \delta^a_m(K^a{}_a - K^{a+1}{}_{a+1}) \qquad a = 1, \ldots, n-1 \\ H_n &= -(K^1{}_1 + K^2{}_2 + K^3{}_3) + K^n{}_n\,. \end{aligned} \tag{2.67}$$

The metric G_{ij} has the universal form

$$\boldsymbol{G}^{+++} = \boldsymbol{I}_D - \frac{1}{2}\Xi_D \text{ and its inverse } \boldsymbol{G}^{-1}_{+++} = \boldsymbol{I}_D - \frac{1}{D-2}\Xi_D\,. \tag{2.68}$$

The Weyl transformations corresponding to simple roots in the gravity line induce an exchange of neighbouring components of the diagonal metric. The Weyl transformation for the simple root α_n induces the changes

$$\begin{aligned} K'^a{}_a &= K^a{}_a \quad a = 1,2,3 \\ K'^a{}_a &= K^a{}_a - H_n \quad a = 4, \ldots, n-1 \qquad K'^n{}_n = K^n{}_n - 2H_n\,. \end{aligned} \tag{2.69}$$

On the p^a the effect of the Weyl transformation is given by

$$\begin{aligned} p'^a &= p^a + (p^4 + \cdots + p^{n-1}) + 2p^n \qquad a = 1,2,3 \\ p'^a &= p^a \qquad a = 4, \ldots, n-1 \\ p'^n &= -p^n - (p^4 + \cdots + p^{n-1})\,. \end{aligned} \tag{2.70}$$

Finally, we point out that the computation of the Weyl reflections generated by the simple roots as well as the outer automorphisms can be performed similarly for all the other simple G^{+++}. From the Dynkin diagram and from table 1, one isolates the SL(D) gravity line and completes the Cartan subalgebra by the required number of dilatons. Weyl reflections corresponding to the simple roots of the gravity line always interchange neighbouring diagonal elements of the metric. One verifies that in the space of the fields $p^i = \{p^a, \Phi^u\}$, the metric G_{ij}^{+++} has always the universal form eq. (2.19).

3. Dimensional reduction revisited

In this section we shall show how the properties of dimensional reduction of 'maximally oxidised' theories [17, 13] can be understood in terms of the general framework laid out in section 2. We consider the theories of gravity in the highest available dimension D, possibly coupled to dilatons and to suitable n-forms with well-chosen coupling to the dilatons, which exhibit, upon dimensional reduction to three space-time dimensions, a symmetry under a simple Lie group $\mathcal{G}$ in its maximally non-compact form. For any simple Lie group $\mathcal{G}$, there exists such a maximally oxidised theory described by a lagrangian $\mathcal{L}_{\mathcal{G}}$.

We first present a review, which follows [13], of the essential features of dimensional reduction from D to three space-time dimensions. We then show how, for a special class of subgroups $\mathcal{G}^{(k)}$ in $\mathcal{G}^{+++}$, group invariant metrics $\boldsymbol{G^{(k)}}$ emerge from dimensional reduction on a k-torus, with $D \geq k \geq 1$. These generalise to all Lie groups the metrics obtained in eq. (2.42) for the E-sequence of E_8^{+++} subgroups. The third and crucial part of the section is dedicated to the formulation of Weyl-preserving embedding equations relative to $\mathcal{G}^{(k)} \subset \mathcal{G}^{+++}$. These play an important role in the following sections.

Dimensional reduction to $D-k$ dimensions is equivalent to compactification on a k-torus of vanishing radii, so as to eliminate the massive Kaluza-Klein excitations. Note that this does not mean that the metric components $g_{\alpha\beta}$ ($\alpha, \beta = D, D-1, \ldots, D-k+1$) of the k-torus are small but rather that the coordinates y^α of the torus metric $ds^2 = g_{\alpha\beta} dy^\alpha dy^\beta$ are periodic with vanishing small periodicity λ. The insensitivity to massive Kaluza-Klein modes implies that the dimensional reduced theory has a symmetry under the deformation group of the k-torus, namely GL(k).

The dimensional reduction from the k-torus is performed step by step so as to remain in the Einstein frame in the remaining non-compact dimensions with a standard kinetic term for the resulting scalar. When k reaches $D-3$, the GL($D-3$) symmetry group is, for the theories defined by $\mathcal{L}_{\mathcal{G}}$, enhanced to a simple non-compact Lie group $\mathcal{G}$. All the original fields of the theory are transmuted to scalars and these form a non-linear realisation of the group on the coset space $\mathcal{G}/\mathcal{H}$ where $\mathcal{H}$ is the maximal compact subgroup of $\mathcal{G}$. Examples of lagrangians $\mathcal{L}_{\mathcal{G}}$ are eqs. (2.41), (2.64), (2.65) and gravity itself, which yield at $D=3$ respectively $\mathcal{G} = E_8$, $\mathcal{G} = D_{D-2}$, $\mathcal{G} = B_{D-2}$ and A_{D-3}.

The group $\mathcal{G}$ is generated in the following way. After k steps one gets, in addition to the dilatons Φ^u, $u = 1, 2, \ldots, q$ originally present, k scalars $\phi^{\hat{a}}$ that parametrise the radii of the k-torus. One also obtains scalars arising from dimensional reduction of the 2-forms $F_{\mu\nu}$ generated at each step from the Kaluza- Klein reduction of gravity. More scalars similarly arise after $n-1$ steps from n-forms present in the original lagrangian and, after $D-n-1$ steps, one also gets scalars by dualising the n-forms. When k reaches $D-3$, all fields can be dualised to scalars. The final result can be described by a lagrangian

$$\mathcal{L} = R - \frac{1}{2}\partial_\mu \varphi^{\hat{\imath}} \delta_{\hat{\imath}\hat{\jmath}} \partial^\mu \varphi^{\hat{\jmath}} - \frac{1}{2}\sum_{\vec{\alpha}} e^{\alpha_{\hat{\imath}}\varphi^{\hat{\imath}}} \partial_\mu \chi^{\vec{\alpha}} \partial^\mu \chi^{\vec{\alpha}} + \cdots , \tag{3.1}$$

where

$$\varphi^{\hat{\imath}} = \{\phi^{\hat{a}}, \Phi^u\} \qquad \hat{a} = D, \ldots, 4 \quad u = 1, \ldots, q\,. \tag{3.2}$$

Here the $\vec{\alpha}$ are euclidean vectors with $D-3+q$ components $\alpha_{\hat{\imath}}$ and they are used to label the additional scalars $\chi^{\vec{\alpha}}$.

The scalar fields in the lagrangian eq. (3.1) form a non-linear realisation of a maximally non-compact Lie group $\mathcal{G}$ in the coset space $\mathcal{G}/\mathcal{H}$ parametrised by

$$g = \sum_{\vec{\alpha}} e^{-\chi^{\vec{\alpha}} E_{\vec{\alpha}}} e^{-\frac{1}{2}\varphi^{\hat{\imath}} H_{\hat{\imath}}}\,. \tag{3.3}$$

The vectors $\vec{\alpha}$ are the positive roots of $\mathcal{G}$ and the Cartan generators $H_{\hat{\imath}}$ are normalised by

$$\mathrm{Tr}(H_{\hat{\imath}} H_{\hat{\jmath}}) = 2\delta_{\hat{\imath}\hat{\jmath}}\,. \tag{3.4}$$

We write the $H_{\hat{\imath}}$-basis as

$$H_{\hat{\imath}} = \{H_{\hat{a}}, R_{\hat{u}}\} \quad \hat{a} = D, \ldots, 4 \quad u = 1, \ldots, q \tag{3.5}$$

where $R_{\hat{u}}$ are the dilaton generators in this new basis.

The group $\mathcal{G}$ is a simple Lie group. This follows from the structure of the simple roots. We now recall in some detail how these arise. First consider the scalars obtained from gravity alone. At the first step of the dimensional reduction, after Weyl rescaling to the Einstein frame and after rescaling the ϕ^D scalar to get the canonical normalisation of the kinetic term, the Einstein action becomes

$$\begin{aligned}\int d^D x \sqrt{-g^{(D)}} R^{(D)} = \int d^{D-1}x \sqrt{-g^{(D-1)}} \Big(& R^{(D-1)} - \frac{1}{2}\partial_\mu \phi^D \partial^\mu \phi^D - \\ & -\frac{1}{4} e^{-2(D-2)\beta_{D-1}\phi^D} F_{\mu\nu}F^{\mu\nu}\Big)\,, \end{aligned} \tag{3.6}$$

where

$$\beta_D = \frac{1}{\sqrt{2(D-1)(D-2)}}\,. \tag{3.7}$$

The further steps proceed similarly and simple roots arise at the first reduction following the appearance of a 2-form. In a sense, this is the 'fastest' way to obtain a $\chi^{\vec{\alpha}}$ scalar in the step by step dimensional reduction process. Reducing $D-3$ dimensions, one gets in this way $D-4$ $\chi^{\vec{\alpha}}$ scalars and the corresponding $\vec{\alpha}$ are the simple roots of $\mathrm{SL}(D-3) = A_{D-4}$. The components of these simple roots in the $(D-3+q)$ dimensional orthonormal euclidean basis defined by eq. (3.2) are

$$\alpha_{(\kappa)\hat{\imath}} = (\underbrace{0, \ldots, 2(D-\kappa-3)\beta_{D-\kappa-1}, -2(D-\kappa-1)\beta_{D-\kappa}, 0, \ldots, 0}_{k=D-3 \text{ terms}}; \underbrace{0, \ldots, 0}_{q \text{ terms}})\,, \tag{3.8}$$

where $\kappa = 1, 2, \ldots, D-4$ and there are $\kappa - 1 + q$ zeros on the right.

The Dynkin diagram of $\mathrm{SL}(D-3) = A_{D-4}$ and the $D-3$ Cartan generators $H_{\hat{a}}$ parametrised by the $\phi^{\hat{a}}$ extend the subgroup $\mathrm{SL}(D-3)$ to the full deformation group $\mathrm{GL}(D-3)$. Note that we have omitted from the above consideration the scalars that arise by dualising the 'graviphotons' $F_{\mu\nu}$ present in three dimensions. The reason is that they do not generate simple roots, except when the original lagrangian $\mathcal{L}_{\mathcal{G}}$ contains only gravity, in which case one dualised graviphoton does.

When suitable n-forms and q dilatons are present in the uncompactified theory, the enhancement of symmetry from $\mathrm{GL}(D-3)$ to $\mathcal{G}$ appears through $q+1$ simple roots attached to the Dynkin diagram of $\mathrm{SL}(D-3)$. The Dynkin diagram of $\mathcal{G}$ is the part of the diagram of $\mathcal{G}^{+++}$ in figure 1 and figure 2 which sits on the right of the dashed line cutting off the three first nodes. Its $\mathrm{SL}(D-3)$ subgroup is the corresponding part of the gravity line $\mathrm{SL}(D)$ in $\mathcal{G}^{+++}$. A deeper relation between these two algebras in the framework of maximally oxidised theories will emerge later in this section.

Let us first illustrate how the general properties of dimensional reduction are realised for the 11-dimensional M-theory, eq. (2.41), and for the bosonic case, eq. (2.64).

(*a*) *M-theory.* The additional simple root is due to the 'electric' 4-form that gets attached to the gravity line after 4 steps. Its components in the orthonormal euclidean basis are

$$\alpha_{(e)\hat{\imath}} = 2(D-5)(\underbrace{0, \dots, \beta_{D-3}, \beta_{D-2}, \beta_{D-1}}_{k=D-3 \text{ terms}}) \,. \tag{3.9}$$

The gravity line has $D-4=7$ nodes and the group $\mathrm{GL}(8)$ is enhanced to the simple group E_8.

(*b*) *The bosonic theory.* The first additional simple root is due to the 'electric' 3-form. It gets attached to the gravity chain after three steps. The second one stems from its 'magnetic' dual $(D-3)$-form and joins the gravity line in three dimensions. The electric and magnetic simple roots components in the orthonormal euclidean basis are

$$\alpha_{(e)\hat{\imath}} = (\underbrace{0, \dots, 2(D-4)\beta_{D-2}, 2(D-4)\beta_{D-1}}_{k=D-3 \text{ terms}}; -l) \,, \tag{3.10}$$

$$\alpha_{(m)\hat{\imath}} = (\underbrace{0, 4\beta_4, \dots, 4\beta_{D-2}, 4\beta_{D-1}}_{k=D-3 \text{ terms}}; l) \,. \tag{3.11}$$

The gravity line has $D-4$ nodes and the group $\mathrm{GL}(D-3)$ is enhanced to the simple group D_{D-2}.

We now make precise the relation of dimensional reduction with the general construction of section 2 and derive the group invariant metrics $\boldsymbol{G^{(k)}}$. To this effect, we relate the fields $\varphi^{\hat{\imath}} = \{\phi^{\hat{a}}, \Phi^u\}$ obtained in the dimensional reduction to the fields $p^i = \{p^a, \Phi^u\}$ in D-dimensional space-time. This amounts essentially to 'undo' the reduction process and to consider, in D dimensions, the moduli of a diagonal $D-3$-torus $p^\alpha, g_{\alpha\alpha} = \exp(2p^\alpha)\eta_{\alpha\alpha}$, for $\alpha = D, D-1, \dots, 4$, and diagonal metric fields in the non-compact dimensions $p^\mu, g_{\mu\mu} = \exp(2p^\mu)\eta_{\mu\mu}$, $\mu = 1,2,3$. Thus we decompose the p^a in two sets, namely $p^a = \{p^\alpha, p^\mu\}$.

We first consider the p^α. When performing a dimensional reduction step by step from D to $D-k$ dimensions, the fields $\phi^{\hat a}$ are shifted from p^a by the Weyl rescalings to the Einstein frame and by the rescaling fixing their kinetic term. Iterating the steps according to eq. (3.6), one gets

$$k \text{ terms} \begin{cases} p^D = -(D-3)\beta_{D-1}\phi^D \\ p^{D-1} = -(D-4)\beta_{D-2}\phi^{D-1} + \beta_{D-1}\phi^D \\ p^{D-2} = -(D-5)\beta_{D-3}\phi^{D-2} + \beta_{D-2}\phi^{D-1} + \beta_{D-1}\phi^D \\ \vdots \\ p^{D-k+1} = -(D-k-2)\beta_{D-k}\phi^{D-k+1} + \cdots + \beta_{D-1}\phi^D \,. \end{cases} \tag{3.12}$$

For $i = D-k+1, \ldots, D+q$, comparing eq. (3.3) with eq. (2.16),

$$p^i L_i \left(\equiv p^\alpha K^\alpha{}_\alpha + \Phi^u R_u\right) = -\frac{1}{2}\varphi^{\hat\imath} H_{\hat\imath} \left(\equiv -\frac{1}{2}\phi^{\hat a} H_{\hat a} - \frac{1}{2}\Phi^u R_{\hat u}\right), \tag{3.13}$$

one has

$$p^\alpha K^\alpha{}_\alpha = -\frac{1}{2}\phi^{\hat a} H_{\hat a}\,, \qquad R_u = -\frac{1}{2} R_{\hat u}\,, \tag{3.14}$$

and we may express the $K^\alpha{}_\alpha$ in terms of the $H_{\hat a}$ as

$$K^\alpha{}_\alpha = H_{\hat a} a^{\hat a}{}_\alpha \,. \tag{3.15}$$

The matrix $\boldsymbol{a} = a^{\hat a}{}_\alpha$ is equal to $(2\boldsymbol{b})^{-1}$ where $\boldsymbol{b}$ is the matrix relating the p^α to the $\phi^{\hat a}$ in eq. (3.12). This yields the triangular matrix

$$\boldsymbol{a} = \begin{bmatrix} \frac{1}{2(D-3)\beta_{D-1}} & 0 & 0 & \cdots & 0 \\ \beta_{D-2} & \frac{1}{2(D-4)\beta_{D-2}} & 0 & \cdots & 0 \\ \beta_{D-3} & \beta_{D-3} & \frac{1}{2(D-5)\beta_{D-3}} & \cdots & 0 \\ \beta_{D-4} & \beta_{D-4} & \beta_{D-4} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ \beta_{D-k} & \beta_{D-k} & \beta_{D-k} & \cdots & \frac{1}{2(D-2-k)\beta_{D-k}} \end{bmatrix} \tag{3.16}$$

For $k = D-3$, the $\mathcal{G}$-invariant metric in the space of the p^i fields can be obtained, as in eq. (2.20), using

$$G^{D-3}_{ij} = \mathrm{Tr}(L_i L_j) = \frac{1}{4}\,\mathrm{Tr}\, R_{\hat u} R_{\hat v} \delta^{\hat u}{}_u \delta^{\hat v}{}_v + \mathrm{Tr}(H_{\hat a} H_{\hat b}) a^{\hat a}{}_\alpha a^{\hat b}{}_\beta \,. \tag{3.17}$$

The normalisation given in eq. (3.4) yields from eq. (3.16)

$$\boldsymbol{G^{(D-3)}} = \frac{1}{2}\boldsymbol{I}_q \oplus (\boldsymbol{I}_{D-3} + \Xi_{D-3})\,, \qquad \boldsymbol{G^{-1}_{(D-3)}} = 2\,\boldsymbol{I}_q \oplus \left(\boldsymbol{I}_{D-3} - \frac{1}{D-2}\Xi_{D-3}\right). \tag{3.18}$$

The metric eq. (3.18) can be generalised for k-torus compactifications with $k \neq D-3$. For $k < D-3$, we shall only consider the sequences of simple groups $\mathcal{G}^{(k)} = E_8^{(k)}, D_{D-2}^{(k)}, B_{D-2}^{(k)}$ of rank k obtained by deleting nodes on the gravity line of $\mathcal{G} = E_8, D_{D-2}, B_{D-2}$, together with the magnetic node attached to it in the D and B series. Such attachment occurs at $k = (D-3)$.[5] The metric for the p^α moduli of the k-torus is obtained as above from eq. (3.16). One gets[6]

$$\boldsymbol{G^{(k)}} = \frac{1}{2}\boldsymbol{I}_q \oplus \left(\boldsymbol{I}_k - \frac{1}{k+2-D}\Xi_k\right), \qquad \boldsymbol{G^{-1}_{(k)}} = 2\,\boldsymbol{I}_q \oplus \left(\boldsymbol{I}_k - \frac{1}{D-2}\Xi_k\right). \tag{3.19}$$

It is important to realise that one may obtain the same equation from the general method of section 2. One embeds $\mathcal{G}^{(k)}$ into $\mathcal{G}^{+++}$ by deleting the nodes m in the Dynkin diagram of $\mathcal{G}^{+++}$ which are not in the Dynkin diagram of $\mathcal{G}^{(k)}$. In the general formalism of section 2, this amounts to equate to zero the field q^m parametrising the Cartan generator H_m in the Chevalley basis, as in the derivation of the metric eq. (2.42) for the E sequence. This method validates the extension of eq. (3.19) for the case where $D \geq k > D-3$. However, it is interesting to obtain this latter extension from dimensional reduction directly. This exercise exhibits the universality of the metrics $\boldsymbol{G^{(k)}}$ up to $\boldsymbol{G^{+++}}$.

We cannot perform dimensional reduction on a k-torus for $k > (D-3)$ in a straightforward way. Gravity does not exist below three space-time dimensions and no new independent scalar could emerge by further compactifications. However the rank of the torus deformation group GL(k) increases up to $k = (D-1)$ when all space dimensions are compactified, and to $k = D$ if time is compactified as well. These deformations add new nodes to the gravity line obtained from dimensional reduction to three dimensions. The resulting Dynkin diagrams define, for $k = (D-2), (D-1)$ and D, the affine Kac-Moody algebra $\mathcal{G}^+$, the lorentzian overextended $\mathcal{G}^{++}$ and the very extended $\mathcal{G}^{+++}$ Kac-Moody algebras. We shall see that all the simple roots of all such Kac-Moody algebras in the L-basis eq. (2.10) are determined by a dimensional regularisation procedure even below three dimensions, and so are all the corresponding metrics eq. (2.18).

We note that, for $k = D-3$, we may compute the roots of the gravity line in the L-basis from eqs. (3.15) and (3.8). One gets, for a given root, two non-vanishing commutators

$$\begin{aligned} [K^\alpha{}_\alpha, E_{\vec{\alpha}}] &= a^{\hat{a}}{}_\alpha [H_{\hat{a}}, E_{\vec{\alpha}}] = +E_{\vec{\alpha}}\,, \\ [K^{\alpha+1}_{\ \alpha+1}, E_{\vec{\alpha}}] &= a^{\hat{a}}{}_{a-\alpha+1}[H_{\hat{a}}, E_{\vec{\alpha}}] = -E_{\vec{\alpha}}\,. \end{aligned} \tag{3.20}$$

These are indeed the components of the roots defined by the commutation relations of the operators $K^\alpha{}_\beta$ given in eq. (2.4). It is easily seen that we still obtain the correct

[5] One could also consider non-simple groups by keeping for instance the magnetic root for $k = D-4$. Indeed, the corresponding enhancement of symmetry appears in the B and D-series by dualising a three form when $k = D-4$, but the extra node gets attached to the gravity line only when k reaches $D-3$. For simplicity we shall restrict the detailed analysis to simple groups $\mathcal{G}^{(k)}$, but the foregoing discussion can be easily extended to such non-simple groups.

[6] Note that $\boldsymbol{G}$ is singular in the affine case $k = D-2$, but that its matrix elements between roots eq. (2.21) are well defined provided they are computed by regularising D as $D+\epsilon$ and taking the limit $\epsilon \to 0$ at the end. The inverse metrics $\boldsymbol{G}^{-1} = (G^{-1})^{ij}$ are well defined for all $k > 2$ and no regularisation is needed to compute scalar products in the affine case. The matrix $\boldsymbol{G}^{-1}$ has, as required by the theory of Kac-Moody algebras, positive eigenvalues for $k \leq (D-3)$, one zero eigenvalue for $k = D-2$, and one negative eigenvalue for $k = D-1$ and $k = D$.

commutation relations for these operators by *defining* $\phi^{\hat{a}}$ for $\hat{a} < 3$ through eq. (3.12) and regularizing β_2, β_1 and β_0. Namely one replaces D by $D + \epsilon$ and takes the limit $\epsilon \to 0$ at the end. All the other simple roots can be computed as above in the L-basis at $k = D - 3$, using for their components in the dilaton space the second equation in eq. (3.14). These roots are not affected by the extensions. For instance the root components in the $H_{\hat{\imath}}$-basis obtained from eqs. (3.9), (3.10) and (3.11) are in the L-basis, for E_8^{+++} and for D_{D-2}^{+++},

$$\alpha_{(e)i} = (\underbrace{0, \ldots, 0, 1, 1, 1,}_{k=D \text{ terms}}) , \tag{3.21}$$

and

$$\alpha_{(e)i} = \left(\underbrace{0, \ldots, 0, 1, 1}_{k=D \text{ terms}} ; \frac{1}{2} l\right) , \tag{3.22}$$

$$\alpha_{(m)i} = \left(\underbrace{0, 0, 0, 0, 1, 1, \ldots, 1}_{k=D \text{ terms}} ; -\frac{1}{2} l\right) . \tag{3.23}$$

The relations eqs. (3.20)–(3.23) are in agreement with the commutation relations in the L-basis listed in the examples of section 2.

Thus the relations eqs. (3.15) with regularised coefficients $a^{\hat{a}}{}_{\alpha}$ remain valid when $k > D - 3$. In addition, extending the normalisation eq.(3.4) to $k > D - 3$, one verifies that the regularised scalar products of all simple roots are well defined. What happens is that the regularised roots acquire one imaginary components when k reaches $D - 1$, thereby switching from the euclidean scalar product $\delta_{\hat{\imath}\hat{\jmath}}$ in eq. (3.4) to a lorentzian one $\eta_{\hat{\imath}\hat{\jmath}}$ for $\mathcal{G}^{++}$ and $\mathcal{G}^{+++}$. The computation eq. (3.17) thus remains valid and the regularised metric eq. (3.19) is recovered in this way for $k > D - 3$.

We now formulate the Weyl-preserving embedding equations that play a key role in unveiling the new symmetries of section 4, and the dualities of section 5. So far we have only considered the k fields p^{α}, $\alpha = D - k + 1, \ldots, D$ but we shall now relate them to the fields p^{μ}, $\mu = 1, \ldots, D - k$ in non-compact dimensions. As pointed out above, the embedding of $\mathcal{G}^{(k)}$ into $\mathcal{G}^{+++}$ may be viewed as the result of deleting nodes m in the Dynkin diagram of $\mathcal{G}^{+++}$. This means that the fields q^m parametrising the Cartan generator H_m in the Chevalley basis are equated to zero. Such embedding relates the $D - k$ fields p^{μ} in the non-compact dimensions to the moduli p^{α} of the compactification torus, and possibly to the dilatons. The relation between fields resulting from the embedding of a subgroup $\mathcal{G}^{(k)}$ obtained by putting Chevalley fields q^m to zero are general and not a specific feature of the dimensional reduction of an oxidised theory. In view of their relevance in sections 4 and 5 we shall thus use the generic labelling of the physical fields $p^i = \{p^a, \Phi^u\}$, and the label (k) in $\mathcal{G}^{(k)}$ will refer in general to the subgroup obtained by deleting the first $D - k$ nodes of the gravity line together with any magnetic node possibly attached to it.

The fields p^i are given in terms of the fields q^m by the relation eq. (2.16), namely $p^a = q^m r_m{}^a$, $\Phi^u = q^m r_m{}^u$. Putting $q^m = 0$ for $m = 1, \ldots, D - k$ and possibly $q^x = 0$ where x labels a magnetic node, one obtains relations between the fields $p^i = \{p^a, \Phi^u\}$ defining the embedding of $\mathcal{G}^{(k)}$ into $\mathcal{G}^{(+++)}$. To derive these relations we shall use the equality between the quadratic forms in the q^m and the p^i variables, namely

$$p^i G^{(k)}_{ij} p^j = p^a G^{(k)}_{ab} p^b + \frac{1}{2}\sum_{u=1}^{q} (\Phi^u)^2 = q^m\, {G_c^{(k)}}_{mn}\, q^n \tag{3.24}$$

where the matrix ${G_c^{(k)}}_{mn}$ for $\mathcal{G}^{(k)}$ is given in eq. (2.15). As seen from eq. (2.18), this equality appears valid only if the matrix $\boldsymbol{r}$ is invertible, that is if the rank of $G^{(k)}_{ij}$ and ${G_c^{(k)}}_{mn}$ are equal. This condition is clearly satisfied if the only deleted nodes belong to the gravity line as each deleted node reduces the rank of both matrices by one unit. It is also satisfied when one deletes the magnetic node attached to the gravity line together with the gravity node to which it is glued. The first case occurs for example in the embedding of the $E^{(k)}$ sequence considered in section 2.2 or for any embedding of $\mathcal{G}^{(k)}$ in $\mathcal{G}^{+++}$ when $k = D-1$, $D-2$ and $D-3$. As mentioned above the second case occurs in the D and B series when $k < D-3$, that is when the magnetic node glued to the fourth root of the gravity line is deleted. The corresponding equation, $q^{D+1} = 0$, yields a relation between the dilaton and the p^a fields which ensures that eq. (3.24) relates expressions with the same number of independent variables.

Consider the embedding $\mathcal{G}^{(k)} \subset \mathcal{G}^{+++}$ and the sequence $q^m = 0$ for $m = 1, \ldots, D-k$ of deleted nodes on the gravity line. It follows from eq. (3.24) that we get $D-k$ relations

$$p^i G^{(s+1)}_{ij} p^j = p^k G^{(s)}_{kl} p^l\,, \tag{3.25}$$

for $s = k, k+1, \ldots, D-1$, and one additional relation between the dilaton and the p^a fields if a magnetic node attached to the gravity line is deleted. From eq. (3.19) we get the explicit relation

$$\begin{aligned}(p^{D-s})^2 + \sum_{a=D-s+1}^{D} (p^a)^2 - \frac{1}{s+3-D}\left(p^{D-s} + \sum_{a=D-s+1}^{D} p^a\right)^2 &= \\ = \sum_{a=D-s+1}^{D} (p^a)^2 - \frac{1}{s+2-D}\left(\sum_{a=D-s+1}^{D} p^a\right)^2 &.\end{aligned} \tag{3.26}$$

Note that it follows from the relation eq. (2.16) between the p^i and the q^m fields, and from footnote 5, that eq. (3.26) is well-defined (after regularisation) even when $s = D-2$ or $D-3$. One gets from eq. (3.26)

$$\left[p^{D-s} - \frac{1}{s+2-D}\sum_{a=D-s+1}^{D} p^a\right]^2 \frac{s+2-D}{s+3-D} = 0\,. \tag{3.27}$$

To obtain the embedding of $\mathcal{G}^{(k)}$ into $\mathcal{G}^{+++}$, it suffices to use the sequence of embedding relations

$$
\begin{array}{rrl}
\mathcal{G}^{++} \subset \mathcal{G}^{+++} \text{ or } (s = D-1): & & p^1 = \sum_{a>1} p^a \\
\mathcal{G}^{+} \subset \mathcal{G}^{++} \text{ or } (s = D-2): & 0 = \sum_{a>2} p^a \;\rightarrow & p^1 = p^2 \\
\mathcal{G}^{D-3} \subset \mathcal{G}^{+}: & -p^3 = \sum_{a>3} p^a \;\rightarrow & p^1 = p^2 \\
\mathcal{G}^{D-4} \subset \mathcal{G}^{D-3}: & -2p^4 = \sum_{a>4} p^a \;\rightarrow & p^3 = p^4 \\
 & & \vdots \\
\mathcal{G}^{(k)} \subset \mathcal{G}^{(k+1)}: & -(D-2-k)p^{D-k} = \sum_{a>D-k} p^a \rightarrow & p^{D-k-1} = p^{D-k}\,,
\end{array}
\tag{3.28}
$$

and possibly the additional relation between the dilaton and the p^a encoded in the relation $\Phi^u = q^m r_m{}^u$. A direct computing of the embedding relations for the p^a using eq. (2.16) yields the relation $p^2 = p^3$ for $\mathcal{G}^{D-3} \subset \mathcal{G}^+$.

The metrics $G^{(k)}_{ij}$ are invariant under the generators of the Weyl group and under the outer automorphisms of the algebra $\mathcal{G}^{(k)}$. The embedding eq. (3.28) guarantees that these transformations are uplifted to the Weyl generators and outer automorphisms of $\mathcal{G}^{+++}$ associated with the roots common to the two algebras. The reader may verify that the embedding equations do indeed determine the Weyl transformations of $\mathcal{G}^{+++}$ from those of $\mathcal{G}^{(k)}$ in all the examples listed in section 2.

We list particular solutions of the embedding equations for the physical fields. Let us first consider the case, studied in the present section, where $\mathcal{G}^{(k)}$ arises from a compactification on a k-torus. A particular solution of eq. (3.28) is obtained by taking all p^μ equal to a constant C. We then get

$$
C = \frac{1}{k+2-D} \sum_{\alpha=D-k+1}^{D} p_\alpha \qquad k \neq D-2\,. \tag{3.29}
$$

This relation will be used in section 5. Another useful relation is the one characterised by the embedding $\mathcal{G}^{++} \subset \mathcal{G}^{+++}$ which is just the first equation in eq. (3.28). For convenience we list it separately here

$$
p^1 = \sum_{a=2}^{D} p^a\,. \tag{3.30}
$$

This relation will be used in section 4.

4. $\mathcal{G}^{+++}$ from Kasner-type solutions

Given a theory invariant under an algebra, it must be the case that any solution of the equations of motion is transformed into another solution under the action of the algebra. Conversely, by carrying out symmetry transformations on a known solution, and checking whether or not the transformed expression still satisfies the equations of motion of the theory, one provides evidence for an underlying symmetry, or concludes the theory does not possess that symmetry. As pointed out before it has been conjectured that E_8^{+++} is a symmetry of M-theory and that D_{24}^{+++} is a symmetry of the bosonic string [12]. In this section we will consider particular classes of solutions not only of the effective actions of these theories, but of all maximally oxidised theories $\mathcal{L}_{\mathcal{G}}$ described in section 3. We will show that given any such solution in $\mathcal{L}_{\mathcal{G}}$, the Weyl group of $\mathcal{G}^{+++}$ transforms it to another one and that these solutions form a representation of the group of Weyl transformations and of outer automorphisms of $\mathcal{G}^{+++}$. This provides evidence for these discrete symmetries and indicates the relevance of the $\mathcal{G}^{+++}$ algebra for the theory defined by $\mathcal{L}_{\mathcal{G}}$.

All these field theories contain gravity and also in some cases a number of scalar fields Φ^u as well as other fields. We will consider solutions for which only the diagonal components of the metric and the scalar fields are non-zero. As explained in sections 2 and 3 one may associate to each such theory a very extended algebras $\mathcal{G}^{+++}$, and taking only these fields to be non-trivial corresponds to selecting a Cartan subalgebra. We have calculated the action of the Weyl group of $\mathcal{G}^{+++}$ on these fields in section 2. As explained there, although one is considering only gravity and scalars fields, one is probing the full Weyl group of $\mathcal{G}^{+++}$.

The non-trivial sector of the theory has an action of the form

$$\int d^D x\sqrt{-g}\left(g^{\mu\nu}R_{\mu\nu}-\frac{1}{2}\sum_{u=1}^{q}g^{\mu\nu}\partial_\mu\Phi^u\partial_\nu\Phi^u\right). \tag{4.1}$$

Varying the metric as well as the dilatons, we find the equations of motion can be written as

$$R_{\mu\nu}=\frac{1}{2}\sum_{u=1}^{q}\partial_\mu\Phi^u\partial_\nu\Phi^u\,,\qquad \partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi^u)=0\quad u=1,\ldots,q\,. \tag{4.2}$$

Putting the off-diagonal components of the metric to zero, we write, following sections 2 and 3, its diagonal components as $g_{aa}=\eta_{aa}e^{2p^a}$, $a=1,\ldots,D$. It is straightforward to calculate the form of the above equation of motion for this diagonal metric. One finds that

$$\begin{aligned}R_{ab}=&\left\{-\partial_a\partial_b\sum_c p^c+\partial_a\partial_b(p^a+p^b)-\sum_c\partial_a p^c\partial_b p^c+\right.\\&\left.+\sum_c\partial_a p^b\partial_b p^c+\sum_c\partial_a p^c\partial_b p^a-2\partial_b p^a\partial_a p^b\right\}e^{-(p^a+p^b)}-\\&-\eta_{ab}\sum_c\eta^{cc}\left\{\partial_c\partial_c p^a+\partial_c p^a\partial_c\sum_d p^d-2\partial_c p^a\partial_c p^c\right\}e^{-2p^c}\\=&\frac{1}{2}e^{-(p^a+p^b)}\sum_{u=1}^{q}\partial_a\Phi^u\partial_b\Phi^u\,,\end{aligned} \tag{4.3}$$

and

$$\sum_{a,b}\eta^{ab}\partial_a\left(\exp\left[\sum_c p^c-p^a-p^b\right]\partial_b\Phi^u\right)=0\,. \tag{4.4}$$

Let us denote the coordinates of our manifold by (v, x^a), $a = 2, \ldots D$ and adopt for p^a the simple dependence $p^a = \tilde{p}^a v$. This yields a line element and dilatons of the form

$$ds^2 = -e^{2\tilde{p}^1 v} d(\tilde{p}^1 v)^2 + \sum_{a=2}^{D} e^{2\tilde{p}^a v} \, (dx^a)^2 \,, \tag{4.5}$$

$$\Phi^u = \tilde{p}^u \, v \quad u = 1, \ldots, q \,. \tag{4.6}$$

We find this is a solution of the above equations of motion if and only if

$$\tilde{p}^1 = \sum_{a=2}^{D} \tilde{p}^a \,, \tag{4.7}$$

$$\sum_{a=2}^{D} (\tilde{p}^a)^2 - \left(\sum_{a=2}^{D} \tilde{p}^a \right)^2 + \frac{1}{2} \sum_{u=1}^{q} (\tilde{p}^u)^2 = 0 \,. \tag{4.8}$$

The first of these constraints arises from the space components R_{aa} equations while the latter arises from the time component R_{vv}. The equation of motion for Φ^u leads to no new constraints.

According to the analysis of section 2, the group parameters $\{\tilde{p}^1 v, \tilde{p}^a v; \tilde{p}^u v\}$ entering the Kasner solutions given by eqs. (4.7) and (4.8) can be viewed as fields parametrising the abelian subalgebra of $\mathcal{G}^{+++}$ in the L-basis. Here $\mathcal{G}^{+++}$ is the very extended Kac-Moody algebra of any Lie group defined in D space-time dimensions and involving q dilatons according to table 1. The subset of fields $p^i = \{\tilde{p}^a v; \tilde{p}^u v\}$, $(a = 2, \ldots, D; u = 1, \ldots, q)$, are the independent moduli of the Kasner solutions. Comparing the Einstein equation (4.8) with eq. (3.19), we see that the moduli p^i satisfy

$$p^i G^{++}_{ij} p^j = 0 \,. \tag{4.9}$$

In this equation we recognise the group invariant metric $\boldsymbol{G}^{++}$ of the overextended algebra $\mathcal{G}^{++}$ restricted to its Cartan subalgebra This metric is invariant under the group $S(\mathcal{G}^{++})$ generated by the Weyl reflections $W_\alpha(\mathcal{G}^{++})$ associated with the simple roots α of $\mathcal{G}^{++}$ and by the outer automorphisms of the Dynkin diagram of $\mathcal{G}^{++}$. It then follows from eq. (4.9) that the moduli space of the Kasner solutions can be decomposed into linear representations of $S(\mathcal{G}^{++})$.

On the other hand, *the Einstein constraint eq. (4.7) is identical to the embedding eq. (3.30)*, and we learn from the general analysis of embeddings given in section 3 that $\mathcal{G}^{++}$ sits in $\mathcal{G}^{+++}$ as the regular embedding obtained by deleting its first gravity node α_1. As discussed there, the generators of $S(\mathcal{G}^{++})$ are identified by this embedding to the corresponding generators of $S(\mathcal{G}^{+++})$, because they leave both metrics $\boldsymbol{G}^{++}$ and $\boldsymbol{G}^{+++}$ invariant. One may verify that eq. (4.7) and eq. (4.9) imply that the moduli $\hat{p}^k = \{\tilde{p}^1 v, \tilde{p}^a v; \tilde{p}^u v\}$ satisfy

$$\begin{aligned} \hat{p}^k G^{+++}_{kl} \hat{p}^l &= \sum_{a=2}^{D} (\tilde{p}^a v)^2 + (\tilde{p}^1 v)^2 - \frac{1}{2} \left(\sum_{a=2}^{D} \tilde{p}^a v + \tilde{p}^1 v \right)^2 + \frac{1}{2} \sum_{u=1}^{q} (\tilde{p}^u v)^2 \,, \\ &= p^i G^{++}_{ij} p^j = 0 \,. \end{aligned} \tag{4.10}$$

Each element of $S(\mathcal{G}^{++})$ acting on the fields p^i is uplifted to an element of $S(\mathcal{G}^{+++})$

through the embedding eq. (4.7). Note that the additional modulus $\tilde{p}^1 v$ is preserved under all the Weyl reflections which simply interchange two p^i in the gravity line of $\mathcal{G}^{++}$. The Weyl generators in $\mathcal{G}^{++}$, associated to electric and magnetic roots, as well as the outer automorphisms, act less trivially and do change the modulus $\tilde{p}^1 v$. Nevertheless they are also uplifted to Weyl generators and automorphisms of $\mathcal{G}^{+++}$, as we illustrate now for M-theory, eq. (2.41), and the bosonic theory, eq. (2.64). A similar analysis can be carried out for all very extended Kac-Moody algebras $\mathcal{G}^{+++}$.

(*a*) *M-theory.* There is no dilaton and the only non-trivial Weyl reflection generator in $E_8{}^{++}$ is $W_{\alpha_{11}}$ where α_{11} is the electric root. It is given by eqs. (2.39) and (2.40) for $a \neq 1$. The Weyl transform p'^1 of p^1 is given in terms of the Weyl transforms p'^a of p^a by eq. (4.7), and we thus get, using eqs. (2.39) and (2.40),

$$\tilde{p}'^1 = \sum_{a=2}^{11} \tilde{p}'^a = \tilde{p}^1 + \frac{1}{3}(\tilde{p}^9 + \tilde{p}^{10} + \tilde{p}^{11})\,. \tag{4.11}$$

As seen from eq. (2.40), this is indeed the correct transformation of the uplifted Weyl reflection $W_{\alpha_{11}}$ in E_8^{+++}, in accordance with the preceding discussion.

(*b*) *The bosonic theory.* There is one dilaton and the non-trivial generators are the Weyl reflections in D_{D-2}^{++} associated with the electric root α_{n-1}, the magnetic root α_n and the outer isomorphism exchanging the roots α_{n-1} and α_{n-2}. We examine each of these transformations.

For $W_{\alpha_{n-1}}$ the Weyl transforms of $\tilde{p}^a v$ and $\Phi = \tilde{p}v$ are given by eqs. (2.52)–(2.54) for $a \neq 1$. Applying eq. (4.7) one finds

$$\tilde{p}'^1 = \sum_{a=2}^{n-1} p'^a = \tilde{p}^1 + \frac{2}{(D-2)}(\tilde{p}^{n-2} + \tilde{p}^{n-1}) + \frac{l}{D-2}\tilde{p}\,, \tag{4.12}$$

which, according to eq. (2.52) is, as expected, the correct transformation of the uplifted Weyl reflection in D_{D-2}^{+++}.

Similarly, for the Weyl generator W_{α_n} corresponding to the magnetic root α_n, the Weyl transforms of $\tilde{p}^a v$ and Φ are read off eqs. (2.56)–(2.58) for $a \neq 1$, and we now get from eq. (4.7)

$$\tilde{p}'^1 = \sum_{a=2}^{n-1} \tilde{p}'^a = \tilde{p}^1 + \frac{D-4}{D-2}(\tilde{p}^5 + \cdots + \tilde{p}^{n-1}) - \frac{l\,(D-4)}{2\,(D-2)}\tilde{p}\,, \tag{4.13}$$

which, as seen from eq. (2.56), is again the correct transformation of the uplifted Weyl reflection in D_{D-2}^{+++}.

Finally the transformations of $\tilde{p}^a v$ and Φ under the outer automorphism are given in eqs. (2.61)–(2.63) ($a \neq 1$) and hence, from eq. (4.7),

$$\tilde{p}'^1 = \sum_{a=2}^{n-1} \tilde{p}'^a = \tilde{p}^1 + \frac{2}{(D-2)}\tilde{p}^{n-1} + \frac{l}{2\,(D-2)}\tilde{p}\,, \tag{4.14}$$

which agrees indeed with the outer automorphism in D_{D-2}^{+++}.

JHEP09(2003)020

To summarise, the moduli p^i of the Kasner solutions form linear representations of the group $S(\mathcal{G}^{++})$ which is the subgroup of $S(\mathcal{G}^{+++})$ generated by all its outer automorphisms and by all its Weyl generators except W_{α_1} associated with the root α_1. This property suggests the construction of an enlarged set of Kasner-like solutions whose moduli do fall in representations of $S(\mathcal{G}^{+++})$. We construct such an enlarged set by considering the following metrics [49] which are analogous to the Kasner metrics but with the role of time and x^2 interchanged. We write

$$ds^2 = -e^{2\tilde{p}^2 v}(dx^2)^2 + e^{2\tilde{p}^1 v} d(\tilde{p}^1 v)^2 + \sum_{a=3}^{D} e^{2\tilde{p}^a v}\,(dx^a)^2\,, \tag{4.15}$$

$$\Phi^u = \tilde{p}^u\, v \qquad u = 1, \ldots, q \tag{4.16}$$

where x^2 is now a time variable and v a space-like one. Eqs. (4.15) and (4.16) solve Einstein equations if and only if,

$$\tilde{p}^2 = \tilde{p}^1 - \sum_{a=3}^{D} \tilde{p}^a\,, \tag{4.17}$$

$$(\tilde{p}^2)^2 + \sum_{a=3}^{D}(\tilde{p}^a)^2 - \left(\tilde{p}^2 + \sum_{a=3}^{D}\tilde{p}^a\right)^2 + \frac{1}{2}\sum_{u=1}^{q}(\tilde{p}^u)^2 = 0\,. \tag{4.18}$$

Similar solutions can be written by singling out any x^a instead of x^2. Eqs. (4.17) and (4.18) are identical to the constraints eqs. (4.7) and (4.8) for the Kasner solutions, although eq. (4.17) can now be obtained from the time component R_{22} (or from any space component $R_{aa}, a > 2$) and eq. (4.18) from the particular space component R_{vv} in eq. (4.3). As previously the equation for the dilaton does not introduce new conditions.

It follows from (4.18) that the moduli of the new solutions fall into representations of a group $\widetilde{S}(\mathcal{G}^{++})$ isomorphic to $S(\mathcal{G}^{++})$. The embedding in $S(\mathcal{G}^{+++})$ defined by eq. (4.17) is however different. The moduli $(\tilde{p}^2 v, \tilde{p}^1 v, \tilde{p}^3 v, \ldots, \tilde{p}^D v; \tilde{p}^u v)$ of the solutions eqs. (4.15) and (4.16) differ from those of the Kasner solutions eqs. (4.5) and (4.6) by the interchange of $\tilde{p}^1$ with the new time modulus $\tilde{p}^2$. This interchange can be obtained from applying the Weyl generator W_{α_1} associated with the root α_1 of $\mathcal{G}^{+++}$. $S(\mathcal{G}^{++})$ and $\widetilde{S}(\mathcal{G}^{++})$ are conjugate in $S(\mathcal{G}^{+++})$ by W_{α_1}. To see this, we first switch p^2 and p^1, and then perform a general transformation of $S(\mathcal{G}^{++})$. The resulting transformation belongs to $\widetilde{S}(\mathcal{G}^{++})$ after relabelling p^1 as p^2. Thus we have

$$\widetilde{S}(\mathcal{G}^{++}) = W_{\alpha_1}^{-1} S(\mathcal{G}^{++}) W_{\alpha_1}\,. \tag{4.19}$$

We stress that W_{α_1} is the only generator of $S(\mathcal{G}^{+++})$ which was not a symmetry of the moduli space of the Kasner solutions. We thus conclude that the enlarged set of Kasner-like solutions constructed in this section span representations of $S(\mathcal{G}^{+++})$.

It is interesting to note that the cosmological billiards [16, 38, 39], which arise as time-dependent solutions of Einstein's equations in the vicinity of a cosmological singularity for the lagrangians $\mathcal{L}_{\mathcal{G}}$ considered in this paper, induce transitions between Kasner solutions spanning linear representations of $S(\mathcal{G}^{++})$. This translates the fact that the billiards walls

JHEP09(2003)020

are invariant under $S(\mathcal{G}^{++})$, a feature which corroborates the possible existence of such a symmetry at a fundamental level. In addition the existence of an embedding which uplifts all the elements of $S(\mathcal{G}^{++})$ to elements of $S(\mathcal{G}^{+++})$ is encoded in Einstein theory, as discussed above. This fact is not immediately apparent in the hamiltonian formalism because the non-trivial Weyl reflections, illustrated in the examples above, affect the time modulus $\tilde{p}^1 v$. As such they imply a redefinition of time which does not affect the hamiltonian constraint. The embedding is the crucial input which allowed us to build a representation of the full $S(\mathcal{G}^{+++})$ from the enlarged set of Kasner-like solutions, providing evidence for a larger symmetry in all oxidised theories.

5. String dualities

It is well-known that symmetries of low energy effective actions of string theories describe symmetries of the perturbative string theories and appear consistent with their symmetries at the non-perturbative level. For instance the effective action of the bosonic string, compactified on a k-torus has a symmetry $O(k,k;R)$ realised non-linearly on the coset $O(k,k;R)/O(k)\times O(k)$ which, in string theory, describes the moduli-space of the compactified string. Such distinct vacua in perturbative string theory should be related to each other if a background-independent non-perturbative formulation were available. In addition the subgroup $O(k,k;Z)$ describes the perturbative symmetry group of generalised T-dualities (for a general review see [50]). The present approach, in which $\mathcal{G}^{+++}$ is viewed as a hidden symmetry of the theory $\mathcal{L}_{\mathcal{G}}$, suggests that the discrete subgroup $S(\mathcal{G}^{(k)})$ describes string dualities in cases where $\mathcal{L}_{\mathcal{G}}$ is the effective action of a string theory (see also references [30, 31, 32]), and that $\mathcal{G}^{(k)}$ is embedded in a larger symmetry group $\mathcal{G}^{+++}$. We shall verify this connection between $S(\mathcal{G}^{(k)})$ and the duality group of string theories, and we shall also check that the embedding of $\mathcal{G}^{(k)}\subset\mathcal{G}^{+++}$, is precisely the one obtained in section 3 by deleting nodes, in accordance with the general construction of section 2.

We shall analyse M-theory from the perspective of an E_8^{+++} symmetry, and the bosonic string theory from that of a D_{24}^{+++} symmetry. More generally we shall define 'string dualities' for the whole D_{D-2}^{+++} series and compare them with those of the B_{D-2}^{+++} series, which is related to the heterotic string.

We organise, as in section 3, the diagonal metric fields $g_{aa}=e^{2p^a}\eta_{aa}$ in two sets: p^α for compact dimensions and p^μ for non-compact ones in Minkowski space. We identify

$$p^\alpha = \ln\left(\frac{L_\alpha}{l_{pl}}\right) \qquad \alpha = D, D-1,\dots,D-k+1 \tag{5.1}$$

$$p^\mu = C \qquad \mu = 1,\dots,D-k \tag{5.2}$$

where C is an arbitrary constant. The embedding of $\mathcal{G}^{(k)}$ in $\mathcal{G}^{+++}$ is given by eq. (3.29).

5.1 M-theory

Consider the M-theory lagrangian eq. (2.41) compactified on a k-torus. The connection with the superstrings is obtained by trading the radius L_{11} in the eleventh dimension and the Planck length l_{pl} for the string coupling constant g_s and the string length l_s. One has

$$L_{11} = g_s l_s \,, \tag{5.3}$$
$$l_{pl} = g_s^{1/3} l_s \,. \tag{5.4}$$

The generators of the discrete symmetries $S(\mathcal{G}^{(k)}) \equiv S(E_8^{(k)}) \subset S(E_8^{+++})$ are given in this case by the generators of the Weyl reflections only. The trivial Weyl generators of $E_8^{(k)}$ simply exchange compactification radii $L_\alpha \leftrightarrow L_\beta$. The non-trivial one is the electric Weyl reflection which acts as

$$p'^\alpha = p^\alpha + \frac{1}{3}(p^{11} + p^{10} + p^9) \qquad \alpha = 12-k, \ldots, 8 \quad (k-3 \text{ terms}\,; k \geq 4) \tag{5.5}$$
$$p'^\alpha = p^\alpha - \frac{2}{3}(p^{11} + p^{10} + p^9) \qquad \alpha = 9, 10, 11 \tag{5.6}$$
$$C' = C + \frac{1}{3}(p^{11} + p^{10} + p^9)\,, \tag{5.7}$$

in accordance with eqs. (2.39) and (2.40). One checks that eq. (5.7) also follows for all k-tori from eqs. (5.5) and (5.6), by using the eq. (3.29) expressing the embedding of $E_8^{(k)}$ into E_8^{+++}.

From eqs. (5.1) and (5.2) as well as eqs. (5.3) and (5.4) that relate gravity to string parameters, one gets

$$\begin{aligned} g'_s &= g_s \frac{l_s^2}{L_{10} L_9}\,, \\ L'_{10} &= \frac{l_s^2}{L_9}\,, \\ L'_9 &= \frac{l_s^2}{L_{10}}\,, \\ L'_\alpha &= L_\alpha \qquad \alpha = 12-k, \ldots, 8 \quad (k-3 \text{ terms}\,; k \geq 4)\,, \end{aligned} \tag{5.8}$$

and

$$C' - C = \ln \frac{l_{pl}}{l'_{pl}}\,. \tag{5.9}$$

The eqs. (5.8) describe a double T-duality [51] for the radii L_{10} and L_9 together with an exchange of these radii. We stress that this derivation of the known relation [30, 31, 32] between T-duality and Weyl reflection for M-theory is done by putting on an equal footing the moduli in compact and non-compact dimensions, as needed if the basic symmetry is the very extended group E_8^{+++}. The bonus is the relation eq. (5.9) which expresses the rescaling of the minkowskian metric in the non-compact dimensions when measured in eleven dimensions, due to the change of the Planck constant under T-duality. This remarkable result bares the signature of $E_{11} = E_8^{+++}$.

5.2 The bosonic string and the D-series

The relation between the string length and the Planck length in the effective action in D dimensions is

$$l_{pl} = g_s^{\frac{2}{D-2}} l_s\,. \tag{5.10}$$

Strictly speaking, this relation only makes sense for $D = 26$ where the quantum bosonic string theory is consistent. However, for sake of generality, we shall leave the dimension D arbitrary and use this relation to define l_s.

The moduli p^i in the L-basis for the D-series are the dilaton Φ and the $p^a = \{p^\alpha, p^\mu = C\}$. When the theory is compactified on a k-torus with radii L_α, p^α is equal to $\ln L_\alpha/l_{pl}$ according to eq. (5.1) and Φ is related to the string coupling constant by

$$\Phi = l \ln g_s\,, \qquad l = \left(\frac{8}{D-2}\right)^{1/2}\,, \tag{5.11}$$

where we have generalised the definition of the string coupling[7] to arbitrary D.

As in the M-theory case, we consider $S(\mathcal{G}^{(k)}) \equiv S(D^{(k)}_{D-2}) \subset S(D^{+++}_{D-2})$. The trivial Weyl reflections in $S(D^{(k)}_{D-2})$ are associated with the gravity line and simply exchange radii. The electric Weyl reflection follows from eqs. (2.52)–(2.54) with $a = \{\alpha, \mu\}$, $\alpha = D-k+1, \dots, D$; $\mu = 1 \dots D-k$. The justification of using eq. (2.52) when $p^\mu = C$ follows from the embedding eq. (3.29) which yields from eqs.(2.53) and (2.54).

$$C' = C + \frac{2}{(D-2)}(p^{D-1} + p^D) + \frac{l}{D-2}\Phi\,, \tag{5.12}$$

in accordance with eq. (2.52) when $a = \mu$. Rewriting all equations in terms of L_α/l_s and g_s, and using eq. (5.10), we recover on the one hand, the double T-duality together with the exchange of the first two radii L_D, L_{D-1}. On the other hand, eq. (5.12) yields the rescaling of the minkowskian metric as in eq. (5.9).

We now consider the magnetic Weyl reflection occurring for $k \geq D-4$.[8] The moduli transformations are given by eqs. (2.56)–(2.58) and may be straightforwardly rewritten in terms of the new variables p^α and C as before. The equation for C is

$$C' = C + \frac{D-4}{D-2}(p^5 + \cdots + p^D) - \frac{l\,(D-4)}{2\,(D-2)}\Phi\,, \tag{5.13}$$

and in terms of the string variables, these read

$$g'_s = g_s^{-1}\frac{L_D L_{D-1}\dots L_5}{l_s^{D-4}}\,, \tag{5.14}$$

$$\frac{L'_\alpha}{l'_s} = \frac{L_\alpha}{l_s}\frac{L_D L_{D-1}\dots L_5}{l_s^{D-4}}g_s^{-2} \qquad \alpha = D-k+1, \dots, 4 \ \ (k \geq D-3) \tag{5.15}$$

$$\frac{L'_\alpha}{l'_s} = \frac{L_\alpha}{l_s} \qquad \alpha = 5, \dots, D\,. \tag{5.16}$$

To ensure that the gravitational constant in four dimensions is invariant under these transformations, l_s must transform as

$$l'_s = l_s g_s^2 \frac{l_s^{D-4}}{L_D L_{D-1}\dots L_5}\,. \tag{5.17}$$

[7] The coefficient l appearing in eq. (5.11) is a consequence of the normalisation for Φ chosen in the effective action eq. (2.64) as compared to the standard normalisation in string theory.

[8] Although the magnetic node gets attached to the gravity line at $k = D-3$, it appears at $k = D-4$ (see footnote 5). The magnetic Weyl reflection at $k = D-4$ is defined as the magnetic Weyl reflection downlifted from $\mathcal{G}^{(k)}, k > D-4$.

Recall that in double T-duality no transformation of l_s is needed to ensure invariance of the gravitational constant. Inserting eq. (5.17) in eqs. (5.15) and (5.16) yields

$$L'_\alpha = L_\alpha \qquad \alpha = D-k+1, \ldots, 4 \;\; (k \geq D-3) \tag{5.18}$$

$$L'_\alpha = L_\alpha \, g_s^2 \frac{l_s^{D-4}}{L_D L_{D-1} \ldots L_5} \qquad \alpha = 5, \ldots, D\,. \tag{5.19}$$

Note that eq. (5.18) implies that the gravitational constant is invariant not only in four, but also in three dimensions. It is precisely the relation eq. (5.18) which ensures that the transformation eq. (5.13) yields the rescaling of the minkowskian metric eq. (5.9), as previously.

We see through eq. (5.14) that the magnetic Weyl transformation reveals in string language the existence of an S-duality for the bosonic string. Such an S-duality has shown up as a symmetry in the coset symmetries of the dimensional reduction to three dimensions of the heterotic and the closed bosonic string effective action [52].

Finally, the outer automorphism is read off eqs. (2.61)–(2.63) and it is easily seen that it describes in the string language the simple T-duality on L_D, namely $L'_D = l_s^2/L_D$; $g'_s = g_s l_s/L_D$ and yields the required rescaling of Minkowski space.

5.3 The heterotic string and the B-series

As in the D-series, we extend the relation between gravity and string parameters eq. (5.10) to any space-time dimension D and use as effective lagrangian the maximally oxidised lagrangian eq.(2.65) that corresponds, when $D = 10$, to an heterotic string with one gauge field only.

It was pointed out in section 2.4 that the magnetic Weyl transformations of the D-series and the B-series coincide. Furthermore the Weyl transformation associated to the short root in the Dynkin diagrams of the B-series yields the same result on the physical fields as the outer automorphism of the D-series. These transformations describe respectively the S-duality and the simple T-duality of the heterotic string and relate in a precise manner the S-duality in the heterotic string and in the bosonic string. As previously, the rescaling of Minkowski space is ensured by the embedding of $B_{D-2}^{(k)}$ into B_{D-2}^{+++}.

6. Conclusion

In this paper we considered the Cartan subalgebra of any very extended algebra $\mathcal{G}^{+++}$ and let the parameters depend on space-time. By selecting a preferred subgroup of $\mathcal{G}^{+++}$ we identified the space-time fields with the diagonal metric fields and possibly scalar fields. One may view this approach as a special case of the more general procedure of taking the full non-linear realisation of $\mathcal{G}^{+++}$ [12, 13]. Namely, one sets to zero all the fields of the full non-linear realisation of $\mathcal{G}^{+++}$ except those associated with the Cartan subalgebra. In the more general setting, the above identification of fields arises naturally. However, in the full non-linear realisation one has an infinite number of fields corresponding to the infinite number of generators in $\mathcal{G}^{+++}$. As there exists no explicit listing of these latter generators it is not clear how to carry out this construction in practice. The advantage of the restriction made in this paper is that one only has a finite number of generators whose behaviour under many properties of the Kac-Moody algebra is known and, as a result, it is possible to perform calculations completely.

We used this set up to test gravity and its coupling in all oxidised theories of gravity and matter. We found evidence for a symmetry under the corresponding very extended algebra $\mathcal{G}^{+++}$, or at least under the group $S(\mathcal{G}^{+++})$ of its Weyl reflections and outer automorphisms. When applied to M-theory or to effective actions of superstring or bosonic string theories, these symmetries also appear in the stringy context, thus providing indications that they are operational at the quantum level as well. These results provide evidence, which corroborates the analysis presented in references [12, 13, 48, 53], for the presence of the very extended symmetries E_8^{+++}, D_{24}^{+++} and A_D^{+++} in M-theory, the closed bosonic string and pure gravity respectively.

We can ask if the appearance of $\mathcal{G}^{+++}$ symmetries points towards the existence of consistent theories of gravity and matter for all $\mathcal{G}$. It is useful to recall the classic case of a non-linear realisation that occurs in a spontaneously broken symmetry in conventional quantum field theory. There, at low energy the dynamics is also given by a non-linear realisation whose degrees of freedom are the Nambu-Goldstone bosons, but the underlying theory possesses many more degrees of freedom and these realise linearly the symmetry. The prototype example is the non-linear realisation of pion dynamics which, as we know now, is accounted for by quarks at a more fundamental level. For those oxidised theories given in eqs. (2.41), (2.64) and (2.65), which can be viewed as effective actions of string theories, the analogy of gravity, dilatons and forms with the Nambu- Goldstone bosons of pion dynamics hiding more fundamental degrees of freedom is borne out by the existence of the huge number of degrees of freedom in string theory itself. Whether the infinite number of fields in the non-linear realisation $\mathcal{G}^{+++}$ could lead to uncovering such new dynamical degrees of freedom and not simply to a reformulation of the known effective actions, is unclear. However, the very fact that the realisation would still be a non-linear one may be an indication that the full theory underlying such degrees of freedom are to be found in a more elaborate realisation of the $\mathcal{G}^{+++}$ symmetry.

The analysis of this paper points towards the existence of very extended symmetries, not only for M-theory and for the bosonic string, but for all oxidised theories. The M-theory quest is generally viewed as the privileged way to reach a unified theory of gravity and matter. From the point of view developed in this paper, the bosonic part of the M-theory effective action eq. (2.41) is just one amongst many sharing the same universal type of symmetry. Even the simplest oxidised action, namely pure gravity, could also be some kind of effective action hiding many fundamental degrees of freedom. Einstein's theory reveals the correct thermodynamical entropy of Schwarzschild and Kerr black holes, but hitherto does not provide an unambiguous derivation of its statistical content. New degrees of freedom are apparently needed to find the quantum states of the black hole. Although string theory does provide such degrees of freedom and explains successfully the entropy of some black holes, the origin of these degrees of freedom when only gravitational quantum numbers are present might well be hidden in the A^{+++} symmetry. In addition,

JHEP09(2003)020

the embedding of superstrings into the bosonic string [43, 44] whose effective action is also an oxidised theory eq. (2.64), suggests that the degrees of freedom hidden in different $\mathcal{G}^{+++}$ may be related to each other.

These considerations motivate the search for structures where the $\mathcal{G}^{+++}$ symmetries could be dynamically realised and which would provide links between different very extended symmetries.

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JHEP09(2003)020

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JHEP09(2003)020